A game theoretic approach to graph problems

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Abstract: We investigate some well known graph theoretic problems from a game theoretic point of view. To coloring and matching problems we associate binary payoff games where the players are the vertices of the graph. Solutions to the graph problems correspond to action profiles of the game, where all players get payoff 1. We show, that there exist rules for the choice of action in the repeated play of these games, that converge to the solution of the graph problems. Although the convergence is slow, this shows, that the problems can be solved with almost no information on the underlying graph.

1 Introduction

Many classical graph problems, that are well observed for the case, that the underlying graph is part of the input become much more challenging if the graph is an existing network and there is no global instance to solve the problem. It is the task of the nodes which are the decision makers to solve the problem, using only information on their neighborhood in the network. Starting with the pioneering work of Linial [Li92] a rich literature of what can be done and what cannot be done using only such local information emerged in the field of distributed computing. However, to the best of the author's knowledge there is no generally accepted definition of local algorithms. Some authors restrict the knowledge of one agent to some small part of the graph near the vertex corresponding to the agent[An07], others allow each vertex only to communicate with its immediate neighbors [LOW08, KMW04]. We propose an approach inspired by the paper of Kearns et al[KSM06] on an experimental study of social network behavior. Their idea is, that the agents corresponding to the nodes of the network graph have no common goal but each of them has a selfish incentive and the solution of the graph problem corresponds to a Nash equilibrium or other suitable solutions of a game that reflects these incentives. In their study the authors investigated the usual vertex coloring problem and the test subjects corresponding to the vertices of the graph got money if they where successful in choosing a color distinct of the colors chosen in the neighborhood in one round of the game. Recently, Chaudhuri et al. [CCJ08] theoretically investigated this game. They could show, that if the number of colors is at least $\Delta + 2$, then with high probability $1 - \delta$ the graph is colored properly within $O(log(n/\delta)$ rounds, where n denotes the number of vertices.
We propose to generalize the idea by investigating for which graph problems it is possible to design such a game, where the players are the vertices of the graph and a solution of the game is a solution for the graph problem. Moreover, the payoff function for each

player should depend only on the actions of the neighbors or be at least easily accessible. Computing pure Nash equilibria of games on graphs can be very hard even if centralized computation is possible [DT07, ZCT08]. We are interested in wether there exist adaption rules for the players such that the repeated play converges to a solution of the game. The information used for the adaption process are only the received payoff and the action of a player in one round. This defines a new concept of local computability of graph problems which reflects the possibility of self organization of large networks without global knowledge.
In the present paper we investigate some coloring and matching problems fitting to the framework. Our results show, that very simple algorithms converge to optimal solutions of the problem, but it may take a lot of time.

2 Preliminaries

Throughout this paper we consider simple finite graphs $G = (V, E)$, where $V = [n] = \{1, ..., n\}$. A *proper k-coloring* of G is a function $c : V \to [k]$ such that for all edges $ij \in E$ we have $c(i) \neq c(j)$. We identify such a coloring with the n dimensional vector $(c(i))_{i\in[n]}$. The smallest number k such that G has a proper k-coloring c is called the chromatic number $\chi(G)$.

Problem 1 (VERTEX COLORING)
Given a graph G and the chromatic number $k = \chi(G)$ compute a proper k-coloring c of G

A well-known extension of this problem is the list coloring problem. Apart from the graph G we are given a *list assignment* $l : V \to 2^{\mathbb{N}}$, where $l(i) \subseteq \mathbb{N}$ denotes the set of admissible colors for the vertex $i \in V$. A proper l- coloring of G is a function $c : V \to \mathbb{N}$ with the property that $c(i) \in l(i)$ for all $i \in V$ and $c(i) \neq c(j)$ for all edges $ij \in E$. G is called l-colorable if there exists a proper l coloring of G.

Problem 2 (VERTEX LIST COLORING)
Given a graph G and a list assignment l such that G is l colorable compute a proper l-coloring c of G

If $l(i) = [k]$ for all vertices $i \in V$ this is equivalent to problem 1.

A *matching* of G is a subset $M \subseteq E$ of edges with the property, that no two edges of M have an end vertex in common. M is *maximal* if all edges outside M have an end vertex in common with an edge in M. A *maximum matching* is a maximal matching with maximum cardinality and a *perfect matching* is a matching M where every vertex $i \in V$ is an end vertex of an edge in M.

Problem 3 (PERFECT MATCHING)
Given a graph G containing a perfect matching compute a perfect matching of G.

Note that while deciding whether a given graph is k-colorable is NP-complete [GJ79], a maximum matching can be found in polynomial time [Ed65].

A finite game $\Gamma = (N, A, u)$ consists of

- A set $N = \{1, ..., n\}$ of players.
- For every player $i \in N$ a set A^i of actions and $A = A^1 \times ... \times A^n$.
- A payoff function $u = (u^i)_{i \in [n]}$ where $u^i : A \to \mathbb{R}$ denotes the payoff function of player i.

An element $a^i \in A^i$ is called action and an element $a = (a^1, ..., a^n) \in A$ is an action profile. The payoff function associates to every possible action profile a payoff for every player. Let S be a subset of players. By (a^{-S}, b^S) we denote the action profile, where each player $i \in S$ chooses action $b^i \in A^i$ and all players $i \notin S$ choose $a^i \in A^i$. A *(pure) Nash equilibrium* of Γ is an action profile $a \in A$ where no player has an intention to deviate, i.e.

$$\forall i \in N \forall b^i \in A^i : u^i(a) \geq u^i(a^{-i}, b^i)$$

The games we investigate in this papers have the property that there are only two possible payoffs 0 and 1. We call such games *binary payoff games*. We interpret a payoff of 1 as a win and a payoff of 0 as a loss. An action $a^i \in A^i$ that ensures a payoff 1 for player i independent of the other player's actions is called a *winning strategy* for player i. An action profile a^S for a subset S of players that ensures a payoff of 1 for all players of S regardless of the actions of $N \setminus S$ is called *cooperative winning strategy (cws)* for the players of S. A subset S of players having a cooperative winning strategy is called *potentially successful*. If the game Γ is repeated infinitely often, the players may adapt there choice of action. Depending on the action of a player and the received payoff in one round of the game Γ a *probabilistic 1-recall learning rule* computes a probability distribution on the set of actions according to which the action for the next round is chosen. If every player uses such a learning rule, this induces a Markov chain on the set of action profiles.

We associate binary payoff games corresponding to the three graph problems stated above. In every case, the players of the game are the vertices of the graph.

COLORING GAME $\Gamma_1 = (N, A, u)$

- $N = V$
- $A^i = [k]$
- $u^i(a) = 1 \;\Leftrightarrow\; a^i \neq a^j$ for all $ij \in E$

That means, every player chooses a color and gets payoff 1 if her color is different from the colors of the neighbors.

This is the game first introduced by Kearns et al. in [KSM06]. A proper k-coloring of the graph corresponds to a Nash equilibrium of the game, since every player gets maximum payoff. If $k \geq \Delta + 1$ where Δ is the maximum degree of G the opposite is also true. To see this, consider a player who receives payoff 0. The neighbors use at most Δ different colors, so the player can choose at least one different color and receive a payoff 1. This shows, that in a Nash equilibrium a every player must get payoff 1, which means that a must be a proper coloring.
In [CCJ08] the authors propose a probabilistic 1-recall learning rule, such that the corresponding Markov chain converges to a proper coloring, given that $k \geq \Delta + 2$.
If $k \leq \Delta$ Nash equilibria do not coincide with proper colorings in general. If G is for instance the complete graph on $V = [n]$ with the edge connecting the vertices 1 and 2 missing, the following coloring is a Nash equilibrium: $a^i = i$ for $1 \leq i \leq n-1$ and $a^n = n-1$. All players apart from $n-1$ and n receive payoff 1, and the last two vertices cannot increase their payoff since all colors appear in their neighborhoods. Nevertheless the graph is $(n-1)$-colorable. If the graph G is the complete bipartite graph $K_{r,r}$ the situation is even worse. There are two proper 2-colorings. But almost every coloring is a Nash equilibrium. As long as both colors appear in both partite sets, every player gets payoff 0 and cannot increase the payoff, since both colors appear in the neighborhood. That means there are $2^{2r} - 4 \cdot 2^r + 6$ colorings only 2 of which correspond to proper colorings.
On the other hand, an action profile a^S for a subset S of players is a cws if and only if all pairs of adjacent vertices in S are colored differently and all players outside S are in different components than the vertices of S. Otherwise, a vertex outside S adjacent o a vertex $i \in S$ could choose color a^i and player i looses. That means in case G is connected, the only potentially successful set of players is the set of all vertices.

The game associated with problem 2 is the following:

LIST COLORING GAME $\Gamma_2 = (N, A, u)$

- $N = V$
- $A^i = l(i)$
- $u^i(a) = 1 \Leftrightarrow a^i \neq a^j$ for all $ij \in E$

Again, every l-coloring of G corresponds to a Nash equilibrium of the game, but the opposite is not true in general. A subset S of players is potentially successful with a cws a^S if every pair (i, j) of adjacent vertices of S is colored differently by a^S and for every edge ij with $i \in S$ and $j \notin S$ we have $a^i \notin l(j)$. We claim that every such partial coloring a^S can be extended to a proper l-coloring of the whole graph G.
If $b \in A$ is any action profile corresponding to a proper l-coloring of G, then (b^{-S}, a^S) is also a proper l-coloring. This is the case because by definition $b^i \neq b^j$ for all edges ij with $i, j \notin S$, $a^i \neq a^j$ for all edges ij with $i, j \in S$ and for all edges ij with $i \in S$ and

$j \notin S$ we have $a^i \neq b^j$ because $a^i \notin l(j)$.

For the perfect matching problem we consider two different games.

FIRST MATCHING GAME $\Gamma_3 = (N, A, u)$

- $N = V$
- $A^i = N(i) = \{j \in V \mid ij \in E\}$
- $u^i(a) = 1 \;\Leftrightarrow\; \exists j \in N(i) : a^i = j \wedge a^j = i$

For an action profile a we consider the set $M(a) = \{ij \in E \mid a^i = j \wedge a^j = i\}$. Obviously $M(a)$ is a matching for all action profiles $a \in A$. An action profile is a Nash equilibrium of Γ_3 if and only if M is a maximal matching, and a set S is potentially successfull, if and only if there exists a matching M such that S is the set of all end vertices of M.

SECOND MATCHING GAME $\Gamma_4 = (N, A, u)$

- $N = V$
- $A^i = N(i) = \{j \in V \mid ij \in E\}$
- $u^i(a) = 1 \;\Leftrightarrow\; \exists j \in N(i) : a^i = j \wedge a^j = i$ and for $k \neq j$ $a^k \neq i$

The set $M(a)$ for an action profile is defined as above. Again a is a Nash equilibrium if and only if $M(a)$ is a maximal matching. However, the only potentially successful sets of players are all vertices of one component of G or all vertices of the union of some components of G, given that G has a perfect matching.

3 Results

We propose the following simple trial and error learning rule for a player i the game Γ_2.

1. Choose equiprobably a random color out of $l(i)$ for the action a^i in the first round of the game.
2. After round t ($t \geq 1$) keep the current color if round t is won, otherwise choose equiprobably a random color out of $l(i)$ for the next round.

Now consider the Markov chain $(X_t)_{t \in \mathbb{N}}$ on the state space A where X_t^i denotes the color chosen by i in round t, if all players apply the trial and error rule.

Theorem 1 *If all players in the game Γ_2 act according to the trial and error rule, with probability 1 there is a time T after which $X_t = c$ for all $t \geq T$, where c is a proper l-coloring of the graph G.*

Proof. For every proper l-coloring c the set $S_c = \{c\} \subseteq A$ is an absorbing subset of the state space of the Markov chain $(X_t)_{t \in \mathbb{N}}$, i.e. once the trajectory of the Markov chain hits the set, it will never leave it. This is obvious, since every player gets payoff 1 if all players choose a color according to c. Since the state space is finite, all we have to show is, that these are the only minimal absorbing subsets of the space state. Assume there is a subset $S \subseteq A$ which is minimal absorbing, i.e. any trajectory of the Markov chain hitting S cannot leave S and S contains no proper subset which is absorbing. Thus, no action profile in S corresponds to a proper l-coloring. Let c be a proper l-coloring and a an action profile in S where a maximum number of vertices chooses a color according to c. Since a is not a proper coloring, there must be an edge ij such that $a^i = a^j$. But then at least one of the two vertices, say i is not colored according to c. Since both players loose in a with positive probability i changes the color to c^i and all other players keep their colors. That means (a^{-i}, c^i) is in S which contradicts the choice of a. Hence, the only minimal absorbing subsets of A are singeltons corresponding to proper l-colorings of G which proves the statement of the theorem. □

Since the game Γ_1 is a special case of Γ_2 the theorem also applies to usual vertex colorings. For the game Γ_4 we consider the the following trial and error rule for a player i:

1. Choose equiprobably a random neighbor out of $N(i)$ for the action a^i in the first round of the game.

2. After round t $t \geq 1$ keep the current choice if round t is won, otherwise choose equiprobably a random neighbor out of $N(i)$ for the next round.

We consider the Markov chain $(X_t)_{t \in \mathbb{N}}$ on the state space A where X_t^i denotes the neighbor chosen by i in round t, if all players apply the trial and error rule

Theorem 2 *If all players in the game Γ_4 on a graph G having a perfect matching M act according to the trial and error rule, with probability 1 there is a time T after which $X_t = b$ for all $t \geq T$, where the set $M(b) = \{ij \in E \mid b^i = j \wedge b^j = i\}$ is a perfect matching of the graph G.*

Proof. For every action profile b where $M(b)$ is a perfect matching the set $\{b\}$ is absorbing, because all players get payoff 1 and will not change their actions. We argue that these singletons are the only minimal absorbing subsets of A. Suppose there is a different minimal absorbing subset S, then S cannot contain an action profile b where $M(b)$ is a perfect matching. For a perfect matching M let $b \in A$ be the action profile with $b^i = j \;\Leftrightarrow\; ij \in M$ Let a be an element of S with a maximum number of players i with $a^i = b^i$. Since $M(a)$ is no perfect matching there is an edge ij such that $a^i = j$ and $a^j \neq i$. Thus both players get payoff 0 in a and may change their actions in the next round. For at least one of both players, say i $a^i \neq b^i$. But then with positive probability the action profile in the next round is $(a^{-i}, b^i) \in S$ which contradicts the choice of S. Thus, the only minimal absorbing subsets of the state space are singletons corresponding to perfect matchings, which proves the statement of the theorem. □

Remark
If all players in the game Γ_4 on a connected graph G without a perfect matching M act according to the trial and error rule, with probability 1 the play of no player will converge to a constant play.

Theorem 3 *If all players in the game Γ_3 on a graph G without isolated vertices act according to the trial and error rule, with probability 1 there is a subset S of players and a time T after which $X_t^S = b^S$ for all $t \geq T$, where the set $M(b) = \{ij \in E \mid b^i = j \wedge b^j = i\}$ is a maximal matching of the graph G and S is the set of all end vertices of $M(b)$.*

The proof of Theorem 3 is similar to that of Theorem 2 and is omitted here.

4 Concluding Remarks

The results of the paper show that graph problems can be solved using learning algorithms in suitable binary payoff games. Morover, the information on the structure of the graph, needed by the players of the game is very little. In fact, they need nothing more than the own payoff and therefore do not even have to be able to observe the actions of the neighbors directly. On the other hand, convergence needs a lot of time. But since the coloring problems are NP-complete, efficient algorithms were not to be expected. The results seem to indicate, that the simple trial and error rules lead to convergence to a desirable outcomes like Nash equilibria or cooperative winning strategies in any binary payoff game. That this is not the case was shown by an example of Hart and Mas-Colell (Theorem 1 in [HM06]).

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