Encoding monotonic multiset preferences using CI-nets[1]

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Abstract: CP-nets and their variants constitute one of the main AI approaches for specifying and reasoning about preferences. CI-nets, in particular, are a CP-inspired formalism for representing ordinal preferences over sets of goods, which are typically monotonic. Considering also that goods often come in multisets rather than sets, a natural question is whether CI-nets can be used more or less directly to encode preferences over multisets. We here provide some initial ideas about this by first presenting a straight-forward generalisation of CI-nets to multisets with bounded multiplicities, which we show can be efficiently reduced to CI-nets. Second, we sketch a proposal for a further generalisation which allows for encoding preferences over multisets with unbounded multiplicities, yet characterise reasoning in this framework in terms of the first. We finally show a potential use of our generalisation of CI-nets for personalization in a recent system for evidence aggregation.

Keywords: Multiset preferences, CI-nets, evidence aggregation

1 Introduction

CI-nets [BEL09] are part of several languages for specifying and reasoning about preferences that are inspired by CP-nets [Bo04]. These languages have in common that assertions regarding preferences are interpreted via the “ceteris-paribus” (“all remaining things being equal”) semantics. I.e. “A is preferred to B” is interpreted as shorthand for “A is preferred to B, ceteris paribus”. This allows the formulation of an “operational semantics” in terms of “worsening flips” for verifying statements regarding preferences computationally. CI-nets distinguishing feature is that they are tailored to ordinal preferences over sets of goods. These are also typically monotonic, i.e. more goods are usually preferred to less goods.

Also taking in account the fact that more often than not goods come in multisets rather than sets, a natural question is whether CI-nets can be easily generalised to specify and reason about preferences over multisets as well as sets of goods. We here present ideas on how to generalise CI-nets to deal with what we identify as the two main differences of preferences over multisets and preferences over sets of goods. The first of the differences is obviously that, while preferences over sets involve comparing different combinations of a fixed number of elements (namely one of each item), when considering multiset preferences also the multiplicity of the items needs to be taken in account. So, for example, while in the set scenario preferring apples over oranges always is interpreted as “irrespective of the number of apples and oranges”, in the multiset scenario it is possible to say, for instance, that one prefers having an apple over an orange if one doesn’t already have any apples, but

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one prefers having an orange over some number (say, up up to three) apples if one already has some (e.g. two or more) apples.

A slightly more subtle issue is that, while when talking about preferences over sets there is a natural limit to the number of items one is considering (namely, one of each), in the case of preferences over multisets it is often the case that it is artificial to impose any a-priori upper bound on the multiplicity of the items. For example, when one says that one prefers having an apple and an orange over say even up to three pears, this also means that one prefers having two apples and two oranges over three pears, three apples and one orange over three pears, etc. If one is using the preferences as a guide as to what choice to take regarding some outcome, e.g. choosing between different baskets of fruits, then the upper bound of apples, oranges, and pears is given by the "evaluation context" (in this case, the upper bound of the fruits in the baskets that are available), but is not part of the preference relation per se. I.e., the same preference relation should be of use when considering a different "evaluation context", e.g. a different set of fruit baskets.

Concretely, in this work we first present a simple generalisation of CI-nets to multisets with fixed multiplicities (Section 3.1). We call the resulting framework C^mI-nets ("m" stands for "multiset"). We show that reasoning on C^mI-nets can be efficiently reduced to reasoning on CI-nets (Section 3.2). We then sketch a proposal for a further generalisation, $C^{\aleph_0}I$-nets (Section 4.1), which allows for encoding preferences over multisets with unbounded multiplicities (hence, the $\aleph_O$ in $C^{\aleph_0}I$-nets), yet characterise reasoning in this framework in terms of reasoning about C^mI-nets (Section 4.2). The result is that at least a restricted form of reasoning on $C^{\aleph_0}I$-nets, which we call "confined reasoning", can ultimately be efficiently reduced to reasoning on CI-nets. Hence, computational procedures and systems for CI-nets [SBH16] can also be used or easily adapted to the multiset scenario.

To further motivate our generalization of CI-nets we give an example of its use in the context of a recent system for the aggregation of evidence from clinical trials [HW12]. We show how $C^{\aleph_0}I$-nets can be applied to order the evidence, which is then subject to further critical analysis by the system, based on personalized criteria (Section 5)[3].

2 Background: CI-nets.

We begin by introducing CI-nets following [BEL09]. Let O be a finite set of objects, items or goods. A *CI-net* on O consists in a set of *CI-statements*: expressions of the form $S^+, S^- : S_1 \rhd S_2$ with S^+, S^-, S_1, S_2 pairwise disjoint subsets of O, $S_1 \neq \emptyset$, $S_2 \neq \emptyset$. The intended meaning is: "if I have all the items in S^+ and none of those in S^-, I prefer obtaining all items in S_1 to obtaining all those in S_2, ceteris paribus". S^+ and S^- are the positive and negative precondition respectively; if they are both empty we write $S_1 \rhd S_2$. The formal semantics of CI-nets on O are given in terms of monotonic preference relations over 2^O. A (strict) *preference relation* is a strict partial order $>$ over 2^O; it is *monotonic* if $S_a \supset S_b$ entails $S_a > S_b$ (S_a "*dominates*" S_b) for any $S_a, S_b \in 2^O$. The preference relation $>$ *satisfies* $S^+, S^- : S_1 \rhd S_2$ if for every $S' \subseteq (O \setminus (S^+ \cup S^- \cup S_1 \cup S_2))$, $(S' \cup S^+ \cup S_1) > (S' \cup S^+ \cup S_2)$.

[3] A longer version of this work is available on arXiv.org [DH16].

A preference relation over 2^O *satisfies a CI-net* $\mathcal{N}$ if it satisfies each CI-statement in $\mathcal{N}$ and is monotonic. A CI-net $\mathcal{N}$ is *satisfiable* if there is a preference relation satisfying $\mathcal{N}$. Our main interest is in the *induced preference relation*, denoted $>_{\mathcal{N}}$. If $\mathcal{N}$ is satisfiable, this is the smallest preference relation satisfying $\mathcal{N}$.

An alternative operational semantics of CI-nets is given in terms of sequences of worsening flips. Let $\mathcal{N}$ be a CI-net on O, and $S_a, S_b \subseteq O$. Then $S_a \rightsquigarrow S_b$ is a *worsening flip* w.r.t $\mathcal{N}$ if either (i) $S_a \supset S_b$ ($\supset$ flip) or (ii) there is an $S^+, S^- : S_1 \rhd S_2 \in \mathcal{N}$ and $S' \subseteq (O \setminus (S^+ \cup S^- \cup S_1 \cup S_2))$ s.t. $S_a = (S' \cup S^+ \cup S_1)$ and $S_b = (S' \cup S^+ \cup S_2)$ (CI flip). See [BEL09] for a more operational version of the latter condition. We denote there being a sequence of worsening flips from S_a to S_b w.r.t. $\mathcal{N}$ as $S_a \hookrightarrow_{\mathcal{N}} S_b$ and say that a CI flip is w.r.t. the CI-statement that "justifies" it ; a sequence of flips is then w.r.t. the set of CI-statements that justify the flips in the sequence. Now, if $\mathcal{N}$ is satisfiable, $S_a >_{\mathcal{N}} S_b$ iff $S_a \hookrightarrow_{\mathcal{N}} S_b$. Also, $\mathcal{N}$ is satisfiable iff there is no S_a s.t. $S_a \hookrightarrow_{\mathcal{N}} S_a$.

CI-nets on O can express all monotonic preference relations on 2^O. The flipside is that satisfiability and dominance of CI-nets is PSPACE-complete. Nevertheless, for instance any CI-net with an "acyclic preference graph" (can be checked in PTIME) is satisfiable.

3 Encoding preferences on multisets with fixed multiplicites

3.1 C^mI-nets

We identify a multiset M on a set of objects O via its multiplicity function m_M; $m_M(o)$ is the number of occurrences of $o \in O$ in M. We will often represent such an M in the form $\{(o, m_M(o)) \mid o \in O, m_M(o) \geq 1\}$. We also use standard notation for sets to be interpreted for multisets. The following is a straightforward generalisation of CI-statements tailored to encoding finite multiset preferences, i.e. the multiplicities of the items are fixed.

Definition 1 (C^mI-statements)**.** Let M be a finite multiset on a set of objects O. A C^mI *statement* on M is an expression of the form $M^+, M^- : M_1 \rhd M_2$ where $M^+ \subseteq M$, $M^- \subseteq (M \setminus M^+)$, $M_1, M_2 \subseteq (M \setminus (M^+ \cup M^-))$, $M_1 \neq \emptyset$, $M_2 \neq \emptyset$, and $(M_1 \cap M_2) = \emptyset$.

C^mI-nets consist in a set of C^mI-statements. The semantics of C^mI-nets on M are defined in terms of preference relations over 2^M, with $>$ over 2^M *satisfying a C^mI-statement* $M^+, M^- : M_1 \rhd M_2$ if for every $M' \subseteq (M \setminus (M^+ \cup M^- \cup M_1 \cup M_2))$, we have $(M' \cup M^+ \cup M_1) > (M' \cup M^+ \cup M_2)$ (the conditions on Definition 1 assure that M' is well defined). The notions of a preference relation *satisfying a C^mI-net*, a *C^mI-net being satisfiable*, as well as the *induced preference relation for a C^mI-net $\mathcal{N}$* ($>_{\mathcal{N}}$), are also defined analogously as for CI-nets. It is easy to see that C^mI-nets are indeed a generalisation of CI-nets and that a C^mI-net on M can express all monotonic preference relations on 2^M.

Example 1. Let $\mathcal{N}$ be the C^mI-net on $M = \{(a, 6), (b, 6), (c, 6)\}$ consisting of the following three (numbered, separated by ";") C^mI-statements: **(1)** $\{(a, 1)\} \rhd \{(b, 6), (c, 6), (d, 6)\}$; **(2)** $\{(a, 1)\}, \emptyset : \{(b, 1)\} \rhd \{(c, 3), (d, 3)\}$; **(3)** $\{(a, 3)\}, \{(b, 4)\} : \{(c, 3)\} \rhd \{(d, 3)\}$. In

Example 3 we reduce $\mathcal{N}$ to a CI-net; we can deduce that $\mathcal{N}$ is satisfiable from the fact that the latter has an acyclic dependency graph. The C^mI-statement **3** expresses that if one has three of a but doesn't have four of b (i.e. one has up to two of b), then one prefers having three more of c than three more of d.

Let $M_a, M_b \subseteq M$, then $M_a \rightsquigarrow M_b$ is a *worsening flip* w.r.t. a C^mI-net $\mathcal{N}$ on M if either (i) $M_a \supset M_b$ ($\supset$ flip) or (ii) there is a C^mI-statement $M^+, M^- : M_1 \rhd M_2 \in \mathcal{N}$ and an $M' \subseteq (M \setminus (M^+ \cup M^- \cup M_1 \cup M_2))$ s.t. $M_a = (M' \cup M^+ \cup M_1)$ and $M_b = (M' \cup M^+ \cup M_2)$ (CI flip). The latter condition can be verified as follows: if $\overline{M} = (M \setminus (M^+ \cup M^- \cup M_1 \cup M_2))$, then (i) $(M \setminus (M^- \cup M_2)) \supseteq M_a \supseteq (M_1 \cup M^+)$, (ii) $(M \setminus (M^- \cup M_1)) \supseteq M_b \supseteq (M_2 \cup M^+)$, and (iii) $(\overline{M} \cap M_a) = (\overline{M} \cap M_b)$.We again denote there existing a sequence of worsening flips from M_a to M_b w.r.t. $\mathcal{N}$ as $M_a \hookrightarrow_{\mathcal{N}} M_b$. The following proposition can be proven as Theorems 7 and 8 in [Bo04] (but also follows from the results in Section 3.2).

Proposition 1. Let $\mathcal{N}$ be a satisfiable C^mI-net on M; $M_a, M_b \subseteq M$. Then $M_a >_{\mathcal{N}} M_b$ if and only if $M_a \hookrightarrow_{\mathcal{N}} M_b$. Also, $\mathcal{N}$ is satisfiable iff there is no $M_a \subseteq M$ s.t. $M_a \hookrightarrow_{\mathcal{N}} M_a$.

Example 2. Consider again the C^mI-net $\mathcal{N}$ from Example 1. The following is a sequence of flips from which $\{(a,3),(b,3)\} >_{\mathcal{N}} \{(a,3),(b,2),(d,5)\}$ can be derived: $\{(a,3),(b,3)\}$ $\rightsquigarrow$ $(CI,\mathbf{2})$ $\{(a,3),(b,2),(c,3),(d,3)\} \rightsquigarrow (CI,\mathbf{3})$ $\{(a,3),(b,2),(d,6)\} \rightsquigarrow (\supset)$ $\{(a,3),(b,2),(d,5)\}$. The labels beside the symbols for flips ($\rightsquigarrow$) indicate the type of flip and, for CI flips, the C^mI-statement justifying the flip.

3.2 Reduction of C^mI-nets to CI-nets

We present a reduction of C^mI-nets to CI-nets in Appendix A. Given a multiset M on O and a C^mI-net $\mathcal{N}_M$ on M we there define a CI-net $\mathcal{N}_{S_M}$ on a set S_M and a mapping of every $M' \subseteq M$ to an $\widetilde{M'} \subseteq S_M$ s.t. propositions 2 and 3 (also proved in the appendix) hold.

Proposition 2. Let $\mathcal{N}_M$ be satisfiable and $M_a, M_b \subseteq M$. Then $M_a <_{\mathcal{N}_M} M_b$ iff $\widetilde{M_a} <_{\mathcal{N}_{S_M}} \widetilde{M_b}$.

Proposition 3. $\mathcal{N}_M$ is satisfiable iff $\mathcal{N}_{S_M}$ is satisfiable.

Example 3. The following is the CI-net corresponding to the C^mI-net from Example 1: **(4)** $\{a_6\} \rhd \{b_1,\dots,b_6,c_1,\dots,c_6,d_1,\dots,d_6\}$; **(5)** $\{a_1\},\emptyset : \{b_6\} \rhd \{c_1,c_2,c_3,d_1,d_2,d_3\}$; **(6)** $\{a_1,a_2,a_3\},\{b_3,b_4,b_5,b_6\} : \{c_4,c_5,c_6\} \rhd \{d_1,d_2,d_3\}$; **(7)** $\big\{\{a_i\} \rhd \{a_{i+1}\} \mid 1 \le i \le 5\big\}$; **(8)** $\big\{\{b_i\} \rhd \{b_{i+1}\} \mid 1 \le i \le 5\big\}$; **(9)** $\big\{\{c_i\} \rhd \{c_{i+1}\} \mid 1 \le i \le 5\big\}$; **(10)** $\big\{\{d_i\} \rhd \{d_{i+1}\} \mid 1 \le i \le 5\big\}$. Here $S_M = \{a_1,\dots,a_6,b_1,\dots,b_6,c_1,\dots,c_6\}$. The sequence of flips corresponding to that of Example 2 is: $\{a_1,a_2,a_3,b_1,b_2,b_3\} \dots (CI,\mathbf{8})$ $\{a_1,a_2,a_3,b_1,b_2,b_6\} \rightsquigarrow (CI,\mathbf{5})$ $\{a_1,a_2,a_3,b_1,b_2,c_1,c_2,c_3,d_1,d_2,d_3\} \dots (CI,\mathbf{9-10})$ $\{a_1,a_2,a_3,b_1,b_2,c_4,c_5,c_6,d_4,d_5,d_6\}$ $\rightsquigarrow (CI,\mathbf{6})$ $\{a_1,a_2,a_3,b_1,b_2,d_1,d_2,d_3,d_4,d_5,d_6\} \rightsquigarrow (\supset)$ $\{a_1,a_2,a_3,b_1,b_2,d_1,d_2,d_3,d_4,d_5\}$.

4 Encoding preferences on multisets with arbitrary multiplicities

4.1 $C^{\aleph_0}I$-nets: definition & extensional semantics

Although C^mI-nets are a straightforward generalisation of CI-nets they are somewhat artificial for modelling purposes. This is reflected in the complicated constraints on C^mI-statements (Definition 1) and is a consequence of the restriction to fixed multiplicites (see the discussion in the introduction). $C^{\aleph_0}I$-nets overcome this limitation and provide a more natural representation.

Let again O be a set of objects and $\mathcal{M}_O$ denote all finite multisets on O. $C^{\aleph_0}I$-nets consist of a set of $C^{\aleph_0}I$-statements which have a “precondition” and a “comparison expression”. A *precondition* on o_i $(1 \leq i \leq n)$ is of the form $o_1R_1a_1, \ldots, o_nR_na_n$ where $o_i \in O$, $R_i \in \{\geq, \leq, =\}$, the a_i are integers ≥ 0. A multiset $M' \in \mathcal{M}_O$ *satisfies* the precondition, $M' \models o_1R_1a_1, \ldots, o_nR_na_n$, iff $m_{M'}(o_i)R_ia_i$ *for every* $1 \leq i \leq n$. A precondition P^+ is *satisfiable* if there is some $M' \in \mathcal{M}_O$ s.t. $M' \models P^+$; if P^+ is empty it is satisfied by any multiset. Comparison expressions on the other hand involve *update patterns* of the form $o_1 + +a_1, \ldots, o_n + +a_n$ with each $o_i \in O$ appearing at most once, the $a_i \geq 1$. Again, such an update pattern is defined *on* the objects o_i $(1 \leq i \leq n)$. The *update of a multiset* $M' \in \mathcal{M}_O$ *w.r.t. an update pattern* is $M'[o_1 + +a_1, \ldots, o_n + +a_n] := M''$ where $m_{M''}(o) = m_{M'}(o)$ for $o \in O$ but $o \neq o_i$ for every $1 \leq i \leq n$, and $m_{M''}(o_i) = m_{M'}(o_i) + a_i$ for $1 \leq i \leq n$.

Definition 2 ($C^{\aleph_0}I$-statement)**.** A $C^{\aleph_0}I$*-statement* on O is an expression $P^+ : P_1 \rhd P_2$ where P^+ is a precondition on a $O' \subseteq O$ and P_1, P_2 are update patterns defined on non-empty, disjoint subsets of O. The $C^{\aleph_0}I$-statement is *satisfiable* if the precondition P^+ is.

We often write $\{o_1, \ldots, o_n\}Ta$ for $o_1Ta, \ldots, o_nTa$, $T \in \{\geq, \leq, =, ++\}$. Informally $P^+ : P_1 \rhd P_2$ with $P^+ = \{o_i^+R_i^+a_i^+\}_{1\leq i\leq n_+}$, $P_1 = \{o_j^1 + +a_j^1\}_{1\leq j\leq n_1}$, $P_2 = \{o_k^2 + +a_k^2\}_{1\leq k\leq n_2}$ means: “if I have $R_i^+a_i$ of o_i $(1 \leq i \leq n_+)$, I prefer having a_j^1 more of o_j^1 $(1 \leq j \leq n_1)$, than having a_k^2 more of o_k^2 $(1 \leq k \leq n_2)$, ceteris paribus”.

Definition 3 (Semantics of $C^{\aleph_0}I$-statements)**.** A preference relation $>$ over $\mathcal{M}_O$ *satisfies* a $C^{\aleph_0}I$ statement $P^+ : P_1 \rhd P_2$ if for every $M' \in \mathcal{M}_O$ s.t. $M' \models P^+$, we have $M'[P_1] > M'[P_2]$.

Alternatively, abusing notation we define $P^+ := \{M' \in \mathcal{M}_O \mid M' \models P^+\}$ and for an update pattern $P = \{o_i + +a_i\}_{1\leq i\leq n}$, $M_P := \{(o_i, a_i) \mid 1 \leq i \leq n\}$. Then $>$ satisfies $P^+ : P_1 \rhd P_2$ if for every $M' \in P^+$, we have $(M' \cup M_{P_1}) > (M' \cup M_{P_2})$. Note that if P^+ is unsatisfiable, then the $C^{\aleph_0}I$-statement $P^+ : P_1 \rhd P_2$ is satisfied by any preference relation. The notions of a preference relation *satisfying* a $C^{\aleph_0}I$-net , a $C^{\aleph_0}I$-net being *satisfiable* and the *preference relation induced* by a satisfiable $C^{\aleph_0}I$-net $\mathcal{N}$ $(>_{\mathcal{N}})$ are, again, defined as for CI-nets.

Example 4. The following $C^{\aleph_0}I$-net $\mathcal{N}'$ re-states the C^mI-net from Example 1, but now for arbitrary multisets over $\{a, b, c, d\}$: **(11)** $a + +1 \rhd \{b, c, d\} + +6$; **(12)** $a \geq 1 : b + +1 \rhd \{c, d\} + +3$; **(13)** $a \geq 3, b \leq 2 : c + +3 \rhd d + +3$. We will later be able to show that $\mathcal{N}'$ is also satisfiable. Moreover, also $\{(a, 3), (b3)\} >_{\mathcal{N}'} \{(a, 3), (b, 2), (d, 5)\}$.

The proof of Proposition 5 in [BEL09] which states that CI-nets on O are able to express all monotonic preferences over 2^O can be easily adapted to show that all monotonic preferences over $\mathcal{M}_O$ can be captured via $C^{\aleph_0}I$-nets on O, but does not exclude the need for an infinite number of $C^{\aleph_0}I$-statements. We leave it as an open question whether there is any useful alternative characterisation of the preference relations that can be captured efficiently (hence, also finitely) by $C^{\aleph_0}I$ (and, for that matter, CI) nets.

4.2 Operational semantics & confined reasoning for $C^{\aleph_0}I$-nets

We turn to giving an operational semantics for $C^{\aleph_0}I$-nets in terms of "worsening flips".

Definition 4 (Worsening flips for $C^{\aleph_0}I$-nets)**.** Let $\mathcal{N}$ be a $C^{\aleph_0}I$-net on O and $M_a, M_b \in \mathcal{M}_O$. Then $M_a \rightsquigarrow M_b$ is called a *worsening flip* w.r.t. $\mathcal{N}$ if either (i) $M_a \supset M_b$ ($\supset$ flip), or (ii) there is a $C^{\aleph_0}I$ statement $P^+ : P_1 \rhd P_2 \in \mathcal{N}$ and an $M' \in \mathcal{M}_O$ s.t. $M' \models P^+$, $M_a = M'[P_1]$, and $M_b = M'[P_2]$ (CI flip). Alternatively, $M_a = M' \cup M_{P_1}$, $M_b = M' \cup M_{P_2}$ for an $M' \in P^+$ or operationally: (i) $M_{P_1} \subseteq M_a$, (ii) $M_{P_2} \subseteq M_b$, (iii) $(M_a \setminus M_{P_1}) = (M_b \setminus M_{P_2})$, and (iv) if $M' = (M_a \setminus M_{P_1}) = (M_b \setminus M_{P_2})$, then $M' \in P^+$ (i.e. $M' \models P^+$).

Again, $M_a \hookrightarrow_{\mathcal{N}} M_b$ denotes there exists a sequence of worsening flips from M_a to M_b w.r.t. the $C^{\aleph_0}I$-net $\mathcal{N}$. Proposition 4 can also be proven analogously to Theorems 7,8 in [Bo04].

Proposition 4. Let $\mathcal{N}$ be a satisfiable $C^{\aleph_0}I$-net defined on O, $M_a, M_b \in \mathcal{M}_O$. Then $M_a >_{\mathcal{N}} M_b$ iff $M_a \hookrightarrow_{\mathcal{N}} M_b$. Also, $\mathcal{N}$ is satisfiable iff there is no $M_a \in \mathcal{M}_O$ s.t. $M_a \hookrightarrow_{\mathcal{N}} M_a$.

Note that there is a sequence of flips for $\{(a,3),(b3)\} >_{\mathcal{N}'} \{(a,3),(b,2),(d,5)\}$ where $\mathcal{N}'$ is the $C^{\aleph_0}I$-net from Example 4 that mirrors the sequence of flips in Example 3 and makes use of the $C^{\aleph_0}I$-statements **12** and **13**. Proposition 5, Corollary 1 and 2 give a straightforward characterisation of reasoning about $C^{\aleph_0}I$-nets in terms of "confined reasoning" as defined via $\hookrightarrow_{\mathcal{N},M}$ (for an $M \in \mathcal{M}_O$) in Definition 5.

Definition 5 (Confinement of sequences of worsening flips)**.** Let $M \in \mathcal{M}_O$. A sequence of worsening flips $M_a = M_1 \dots M_n = M_b$ w.r.t. a $C^{\aleph_0}I$-net $\mathcal{N}$ on O is *confined to M* if each flip $M_i \rightsquigarrow M_{i+1}$ (for $1 \leq i < n$) in the sequence is s.t. $M_i, M_{i+1} \subseteq M$. $M_a \hookrightarrow_{\mathcal{N},M} M_b$ denotes there being a sequence of worsening flips from M_a to M_b confined to M. Finally, $\mathcal{N}$ is *c-consistent* w.r.t M if there is no $M_a \subseteq M$ s.t. $M_a \hookrightarrow_{\mathcal{N},M} M_a$.

Proposition 5. Let $\mathcal{N}$ be a $C^{\aleph_0}I$-net on O. $M_a \hookrightarrow_{\mathcal{N}} M_b$ iff $M_a \hookrightarrow_{\mathcal{N},M} M_b$ for an $M \in \mathcal{M}_O$.

Corollary 1. *If $\mathcal{N}$ is satisfiable, then $M_a >_{\mathcal{N}} M_b$ iff $M_a \hookrightarrow_{\mathcal{N},M} M_b$ for an $M \in \mathcal{M}_O$.*

Corollary 2. *$\mathcal{N}$ is satisfiable iff $\mathcal{N}$ is* c-consistent *w.r.t every $M \in \mathcal{M}_O$.*

Now usually one will only be interested in determining whether $M_a >_{\mathcal{N}} M_b$ for some $(M_a, M_b) \in U$ where $U \subseteq \mathcal{M}_O \times \mathcal{M}_O$ is what we called an *evaluation context* in the introduction (in particular, $|U| = 1$). Hence one would also like to know some (small) $M \in \mathcal{M}_O$ s.t. $\hookrightarrow_{\mathcal{N},M}$ *captures* $\hookrightarrow_{\mathcal{N}}$ for U, i.e. $M_a \hookrightarrow_{\mathcal{N}} M_b$ iff $M_a \hookrightarrow_{\mathcal{N},M} M_b$ for every $(M_a, M_b) \in U$.

Example 5. Consider again the $C^{\aleph_0}I$-net $\mathcal{N}'$ from Example 4. This $C^{\aleph_0}I$-net also has the analogue to an acyclic dependency graph for CI-nets. This means that given an initial multiset M_a, lets say $M_a = \{(a,3),(b,3)\}$, one can compute an upper bound on the number of instances of each object one will be able to add to the objects in M_a via worsening flips. Let $\#o$ denote this number for each $o \in O$. Then $\#a = 3$, $\#b = 3 + (\#a * 6) = 21$, $\#c = (\#a*6)+(\#b*3) = 81$, $\#d = (\#a*6)+(\#b*3)+(\#c*3) = 324$, and therefore $\hookrightarrow_{\mathcal{N},M}$, with $M = \{(a,3),(b,21),(c,81),(d,324)\}$, *captures* $\hookrightarrow_{\mathcal{N}}$ for $U = \{(M_a, M') \mid M' \in \mathcal{M}_O\}$.

Example 6 (CI-nets as $C^{\aleph_0}I$-nets)**.** A CI-statement $c = S^+, S^- : S_1 \rhd S_2$ in a CI-net $\mathcal{N}$ can be written as the $C^{\aleph_0}I$-statement $\hat{c} := P^+ : C, P^+ := P_1^+ \cup P_2^+ \cup P_3^+, P_1^+ := \{s^+ = 1 \mid s^+ \in S^+\}$, $P_2^+ := \{s = 0 \mid s^- \in (S^- \cup S_1 \cup S_2)\}$, $P_3^+ := \{s \leq 1 \mid s \in (O \setminus (S^+ \cup S^- \cup S_1 \cup S_2))\}$, and $C := \{s_1{+}{+}1 \mid s_1 \in S_1\} \rhd \{s_2{+}{+}1 \mid s_2 \in S_2\}$. The CI-flips w.r.t. $\mathcal{N}$ and $\hat{\mathcal{N}} := \{\hat{c} \mid c \in \mathcal{N}\}$ are exactly the same and hence $\hookrightarrow_{\hat{\mathcal{N}},O}$ captures $\hookrightarrow_{\hat{\mathcal{N}}}$ for $U = (O \times O)$.

We sketch a translation of confined reasoning about $C^{\aleph_0}I$-nets to C^mI-nets in Appendix B. The C^mI-net from Example 1 is, in fact, the C^mI-net that results when applying this translation for confined reasoning w.r.t. $\{(A,6),(B,6),(C,6)\}$ and the $C^{\aleph_0}I$-net in Example 4. The satisfiability of the $C^{\aleph_0}I$-net in Example 4 follows from Corollary 2 and the fact that the translation of confined reasoning w.r.t. this $C^{\aleph_0}I$-net and *any* $M \in \mathcal{M}_O$ produces a C^mI-net which can be reduced to a CI-net with an acyclic preference graph.

5 Encoding preferences in evidence aggregation

In this Section we show how $C^{\aleph_0}I$-nets can be applied in the context of the system for aggregating evidence presented in [HW12] (see [Wi15] for a recent use). In this system evidence from clinical trials is initially collected in the form of tables of which Table 1 could be an extract (our example is based on Table 3 in [HW12]). Table 1 summarises possible results from meta-analyses ("MA") for patients who have raised pressure in the eye and are at risk of glaucoma. The results of the studies ("Outcome value") have been normalised so that the values are desirable, i.e. they indicate the degree to which the treatment which has fared better in the study presents an improvement (column "Net outcome"; in Table 1 ">","<" means the study speaks for PG, BB resp.). Given the evidence in Table 1, the question is whether PG or BB are better to treat glaucoma.

A first step towards a solution of this problem is determining what sets of evidence items that can be used to argue in favour of the treatments are of more value in terms of preferences over "*benefits*": outcome indicator - normalised outcome value pairs. More to the point, since for methodological reasons (mainly, to avoid bias and for purposes of reuse), preferences need to be determined independently of the available evidence, the preference relation is in terms of *possible* sets of benefits, i.e. all possible sets of pairs of (normalised) outcome indicator-value pairs. Specifying preferences over "benefit sets" allows for a personalised dimension in the decision process, i.e. of considerations which have to do with, e.g., a specific patient or the experience of the medical professional. Other more "objective" elements (like statistical significance - column "Sig" in Table 1) can be incorporated in further stages of the decision process as outlined in [HW12].

ID	Left	Right	Outcome indicator	Outcome value	Net outcome	Sig	Type
e_{01}	PG	BB	change in IOP (SO)	-2.32 (m)	>	no	MA
e_{02}	PG	BB	acceptable IOP (SO)	1.54 (s)	>	yes	MA
e_{03}	PG	BB	respiratory prob	0.9 (s)	>	yes	MA
e_{04}	PG	BB	respiratory prob	0.85 (s)	>	yes	MA
e_{05}	PG	BB	cardio prob	0.82 (s)	>	no	MA
e_{06}	PG	BB	hyperaemia	0.61 (m)	<	yes	MA
e_{07}	PG	BB	drowsiness	0.58 (m)	<	yes	MA
e_{08}	PG	BB	drowsiness	0.71 (m)	<	yes	MA
e_{09}	PG	BB	drowsiness	0.62 (m)	<	yes	MA

Tab. 1: Normalised results of several meta-analysis studies comparing prostaglandin analogue (PG) and beta-blockers (BB) for patients with raised intraocular pressure.

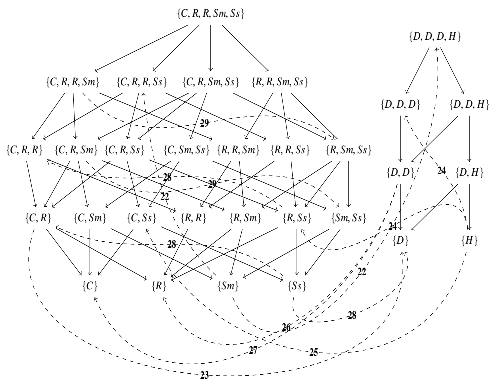

Fig. 1: Graphical representation of the preference relation induced by the $C^m I$-net in Example 8. Solid arcs are obtained by $\supset$, dotted arcs also via CI-flips.

In [HW12] the authors only consider the incorporation of preferences between *sets* of benefits in their system and for this purpose CI-nets would be a natural choice. Also allowing preferences over multisets of benefits to be stated becomes relevant when one considers that -, especially as a result of the use of some abstraction over the outcome indicators and values,- there may be more than one evidence item expressing the same benefit. Example 7 illustrates the use of $C^{\aleph_0}I$-nets for encoding preferences over multisets of benefits such as those appearing in Table 1 but where we introduce a natural abstraction. We consider both “change in IOP” and “acceptable IOP” (“IOP” = interocular pressure) as part of the “significant outcomes” which we denote “SO”; we partition the outcome indicators into “s”, “m”, and “l” standing for a “small”, “medium”, and “large” improvement respectively. The values in parentheses beside the entries for “Outcome indicator” and “Outcome value” show a possible result of applying this abstraction to the results in Table 1.

Example 7. The following is a $C^{\aleph_0}I$-net on the benefits that appear in Table 1. We use C, D, H, R for (*cardio prob*, s), (*drowsiness*, m), (*hyperaemia*, m), and (*respiratory prob*, s) respectively, while $Sm := (SO, m)$ and $Ss := (SO, s)$. $\{a_1, \ldots, a_n\}$## denotes the maximum number of each of $a_1, \ldots, a_n$ in any specific evaluation context. The $C^{\aleph_0}I$ net consists in the statements: **(14)** $Sm + +1 \rhd \{C, D, R, Ss\}$##, $H + +1$; **(15)** $\{C, R\} + +1 \rhd D + +1$; **(16)** $C = 0 : H + +1 \rhd D$##, $\{R, Ss\} + +1$; **(17)** $R = 0 : H + +1 \rhd D$##, $\{C, Ss\} + +1$; **(18)** $C = 0 : D + +2 \rhd R + +1$; **(19)** $R = 0 : D + +2 \rhd C + +1$; **(20)** $Sm = 0 : Ss + +1 \rhd \{C, D, R\} + +1$; **(21)** $Sm \geq 1 : \{C, R\} + +1 \rhd Ss + +1$. $C^{\aleph_0}I$-statement **20**, for example, states that if one does not have any evidence for a modest improvement in the significant outcomes, then evidence for even a small improvement for any of the significant outcomes is preferred to evidence showing an improvement in drowsiness as well as cardio and respiratory problems.

Example 8 gives the encoding of confined reasoning for the $C^{\aleph_0}I$-net of Example 7 w.r.t. all benefits occurring in Table 1. Figure 1 shows the preference relation induced by the

$C^m I$-net in Example 8, but considering only sets of benefits which all result from the *same* treatment according to Table 1.

Example 8. The following is the encoding of confined reasoning for the $C^{\aleph_0} I$-net of Example 7 w.r.t. the multiset $M = \{(C,1),(D,3),(H,1),(R,2),(Sm,1),(Ss,1)\}$. For the encoding we interpret *o*## as the max number of occurrences of o in M. **(22)** $\{(Sm,1)\} \triangleright \{(C,1),(D,3),(H,1),(R,2),(Ss,1)\}$; **(23)** $\{(C,1),(R,1)\} \triangleright \{(D,1)\}$; **(24)** $\emptyset, \{(C,1)\} : \{(H,1)\} \triangleright \{(D,3),(R,1),(Ss,1)\}$; **(25)** $\emptyset, \{(R,2)\} : \{(H,1)\} \triangleright \{(D,3),(C,1),(Ss,1)\}$; **(26)** $\emptyset, \{(C,1)\} : \{(D,2)\} \triangleright \{(R,1)\}$; **(27)** $\emptyset, \{(R,2)\} : \{(D,2)\} \triangleright \{(C,1)\}$; **(28)** $\emptyset, \{(Sm,1)\} : \{(Ss,1)\} \triangleright \{(C,1),(D,1),(R,1)\}$; **(29)** $\{(Sm,1)\}, \emptyset : \{(C,1),(R,1)\} \triangleright \{(Ss,1)\}$.

6 Conclusion & future work

As fas as we are aware this is the first work to present a framework for encoding ordinal multiset preferences, certainly in the context of CI-nets. Our results allow for sound and complete procedures for confined reasoning, the issue of finding a multiset that captures the preference relation w.r.t. a $C^{\aleph_0} I$-net for an evaluation context remaining largely unexplored. As is determining subclasses of $C^{\aleph_0} I$-nets beyond acyclic ones where such a multiset can be found or satisfiability is guaranteed. Complexity issues remain to be explored. Finally, techniques for e.g. CP nets "in practice" [Al15] as well as algorithms and systems for CI-nets [SBH16] can be adapted and optimised for the multiset scenario.

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A Reduction of C^mI-nets to CI-nets

Let M be a multiset on O and a $\mathcal{N}_M$ a C^mI-net on M. We here define a CI-net $\mathcal{N}_{S_M}$ on a set S_M and a mapping of every $M' \subseteq M$ to an $\widetilde{M'} \subseteq S_M$ s.t. propositions 2 and 3 hold.

We start by introducing some notation. Given some $o \in O$ and i, j s.t. $i, j \geq 1$ we define the *forward-generated set of j indexed copies from i* of o as $[o]^F_{i,j} := \{o_i, o_{i+1}, \ldots, o_{i+(j-1)}\}$ and the *backward-generated set of j indexed copies from i* of o as $[o]^B_{i,j} := \{o_i, o_{i-1}, \ldots, o_{i-(j-1)}\}$. If $j = 0$, we define $[o]^F_{i,j} = [o]^B_{i,j} := \emptyset$. Then $S_M := \bigcup\{[o]^F_{1,m_M(o)} \mid o \in O\}$. We call $[o]^F_{1,m_M(o)} = [o]^B_{m_M(o),m_M(o)}$ for $o \in O$ the *set of indexed copies of o in* S_M.

For some $M' \subseteq M$, $*M'$ includes all sets which, for each $o \in O$, have the same *number* of elements from the set of indexed copies of o in S_M as instances of o there are in M'. Formally, we define $*M'$ to be the set $\{S \subseteq S_M \mid |S \cap [o]^F_{1,m_M(o)}| = m_{M'}(o)$ *for every* $o \in O\}$. Clearly, in particular $*M = \{S_M\}$. We also (partially) order the sets in S_M via the order $>_\flat$ defined as the transitive closure of the binary relation $\{(S_1, S_2) \mid (S_1 \cup S_2) \setminus (S_1 \cap S_2) = \{o_i, o_j\}$ *s.t.* $o \in O, o_i \in S_1, o_j \in S_2,$ *and* $j = i + 1\}$. Crucial for our purposes is that there is a unique maximal element $\bigcup\{[o]^F_{1,m_{M'}(o)} \mid o \in O\}$ w.r.t. $>_\flat$ within $*M'$ for every $M' \subseteq M$. We denote this maximal element as $\widetilde{M'}$. Note that in particular $\widetilde{M} = S_M$.

Next we proceed to define for a C^mI-statement $c \in \mathcal{N}_M$ the corresponding CI statement $\widehat{c} \in \mathcal{N}_{S_M}$. Assume c is of the form $M^+, M^- : M_1 \rhd M_2$. For simplicity we write the multiplicity functions of M^+, M^-, M_1, M_2 as m^+, m^-, m_1, m_2 respectively. Then $\widehat{c} := \widehat{M^+}, \widehat{M^-} : \widehat{M_1} \rhd \widehat{M_2}$ where $\widehat{M^+} := \bigcup\{[o]^F_{1,m^+(o)} \mid o \in O\}$, $\widehat{M^-} := \bigcup\{[o]^B_{m_M(o),m^-(o)} \mid o \in O\}$, $\widehat{M_1} := \bigcup\{[o]^B_{m_M(o)-m^-(o),m_1(o)} \mid o \in O\}$, $\widehat{M_2} := \bigcup\{[o]^F_{m^+(o)+1,m_2(o)} \mid o \in O\}$. We denote the set of CI statements corresponding to the $c \in \mathcal{N}_M$ as cs_1. Apart from the CI-statements in cs_1, $\mathcal{N}_{S_M}$ also contains the set of CI-statements $cs_2 := \{o_i \rhd o_j \mid o \in O, 1 \leq i < m_M(o), j = i + 1\}$.

Lemma 1. *If* $M_a \rightsquigarrow M_b$ *is a CI-flip w.r.t* $c \in \mathcal{N}_M$ *then there is a* $S_{M_a} \in *M_a$ *s.t.* $S_{M_a} \rightsquigarrow \widetilde{M_b}$ *is a CI-flip w.r.t.* $\widehat{c} \in \mathcal{N}_{S_M}$*. Also, if there are* $S_{M_a} \in *M_a$, $S_{M_b} \in *M_b$ *s.t.* $S_{M_a} \rightsquigarrow S_{M_b}$ *is a CI-flip w.r.t.* $\widehat{c} \in \mathcal{N}_{S_M}$*, then* $M_a \rightsquigarrow M_b$ *is a CI-flip w.r.t.* $c \in \mathcal{N}_M$*.*

Proof. Let $c = M^+, M^- : M_1 \rhd M_2$. The set of all CI-flips "induced" by $\widehat{c} \in \mathcal{N}_{S_M}$ are of the form $(S' \cup \widehat{M^+} \cup \widehat{M_1}) \rightsquigarrow (S' \cup \widehat{M^+} \cup \widehat{M_2})$ with $\widehat{M^+}, \widehat{M_1}, \widehat{M^+}, \widehat{M_2}$ defined as above and $S' \subseteq \bigcup\{[o]^F_{X_o,Y_o} \mid o \in O\}$ where $X_o = m^+(o)+m_2(o)+1, Y_o = m_M(o)-(m^+(o)+m^-(o)+m_1(o)+m_2(o))$ for each $o \in O$. Then for each M' s.t. $M_a = M' \cup M^+ \cup M_1$, $M_b = M' \cup M^+ \cup M_2$, and $M_a \rightsquigarrow M_b$ is a CI-flip w.r.t. $c \in \mathcal{N}_M$, there is a $S' \subseteq \bigcup\{[o]^F_{X_o,Y_o} \mid o \in O\}$, $S_a \in *M_a$, $S_b \in *M_b$ s.t. $S_a = S' \cup \widehat{M^+} \cup \widehat{M_1}$, $S_b = S' \cup \widehat{M^+} \cup \widehat{M_2}$ and $S_a \rightsquigarrow S_b$ is a CI-flip w.r.t. $\widehat{c} \in \mathcal{N}_{S_M}$. In particular, by construction one can pick $S' = \bigcup\{[o]^F_{X_o,m_{M'}(o)} \mid o \in O\}$ and then $S_b = \widetilde{M_b}$. Also, for $S_a = S' \cup \widehat{M^+} \cup \widehat{M_1}$, $S_b = S' \cup \widehat{M^+} \cup \widehat{M_2}$ s.t. $S_a \rightsquigarrow S_b$ is a CI-flip w.r.t. $\widehat{c} \in \mathcal{N}_{S_M}$, $M_a \rightsquigarrow M_b$ is a CI-flip w.r.t $c \in \mathcal{N}_M$, where $M_a = (M' \cup M^+ \cup M_1)$, $M_b = (M' \cup M^+ \cup M_2)$ and $m_{M'}(o) = |S' \cap [o]^F_{1,m_M(o)}|$ for each $o \in O$. $\square$

Lemma 2. *If $S, S' \subseteq S_M$ and $S >_\flat S'$, then there is a sequence of cs_2 flips from S to S' w.r.t. $\mathcal{N}_{S_M}$.*

Proof. (sketch) This lemma follows from the fact that $>_\flat$ is equivalent to the transitive closure (within S_M) of the CI-flips induced by cs_2. □

Lemma 3. *Let $M_a, M_b \subseteq M$. If $M_a \hookrightarrow_{\mathcal{N}_M} M_b$, then $\widetilde{M_a} \hookrightarrow_{\mathcal{N}_{S_M}} \widetilde{M_b}$. Also, let $S_{M_a} \rightharpoonup_{\mathcal{N}_{S_M}} S_{M_b}$ for some $S_{M_a} \in *M_a$, $S_{M_b} \in *M_b$ denote that there exists a sequence involving at least one non-cs_2 flip w.r.t $\mathcal{N}_{S_M}$. We call such a sequence* non-trivial. *Then, if $S_{M_a} \rightharpoonup_{\mathcal{N}_{S_M}} S_{M_b}$, $M_a \hookrightarrow_{\mathcal{N}_M} M_b$ also is the case.*

Proof. We start by proving by induction on $k \geq 0$, that if there exists a sequence $M_a, \ldots, M_b$ w.r.t. $\mathcal{N}_M$ with k CI flips, then there is a sequence $\widetilde{M_a}, \ldots, \widetilde{M_b}$ w.r.t. $\mathcal{N}_{S_M}$ with k cs_1 flips. The base case ($k = 0$) follows from the fact that if $M_a \supset M_b$ then $\widetilde{M_a} \supset \widetilde{M_b}$ and hence there is a sequence $\widetilde{M_a}, \ldots, \widetilde{M_b}$ consisting only of $\supset$ flips w.r.t. $\mathcal{N}_{S_M}$.

For the inductive case assume that there exists a sequence $M_a, \ldots, M_b$ w.r.t $\mathcal{N}_M$ with $k + 1 \geq 1$ CI flips. Consider the last M_c, M_d in the sequence s.t. $M_c \rightsquigarrow M_d$ is a CI flip. By inductive hypothesis then there is a sequence of flips $\widetilde{M_a}, \ldots, \widetilde{M_c}$ w.r.t. $\mathcal{N}_{S_M}$ with k cs_1 flips. By Lemma 1 there exists a $S_{M_c} \in *M_c$ s.t. $S_{M_c} \rightsquigarrow \widetilde{M_d}$ is a CI-flip w.r.t. $\widehat{c} \in \mathcal{N}_{S_M}$. Hence $\widetilde{M_a}, \ldots, \widetilde{M_c}, \ldots, S_{M_c}, \widetilde{M_d}, \ldots, \widetilde{M_b}$ is a sequence w.r.t. $\mathcal{N}_{S_M}$ with $k + 1$ cs_1 flips. Here $\widetilde{M_c} = S_{M_c}$ or $\widetilde{M_c}, \ldots, S_{M_c}$ is a sequence of cs_2 flips (that such a sequence exists follows from Lemma 2). Also $\widetilde{M_d} = \widetilde{M_b}$ or $\widetilde{M_d}, \ldots, \widetilde{M_b}$ is a sequence consisting only of $\supset$ flips.

We now prove by induction on k, that if there exists a non-trivial sequence $S_{M_a}, \ldots, S_{M_b}$ w.r.t. $\mathcal{N}_{S_M}$ with k cs_1 flips for some $S_{M_a} \in *M_a$, and $S_{M_b} \in *M_b$, then there exists a sequence $M_a, \ldots, M_b$ w.r.t. $\mathcal{N}_M$ with the k CI flips. If $k = 0$, then the sequence $S_{M_a}, \ldots, S_{M_b}$ must have at least one $\supset$ flip, i.e. $S_{M_a} \supset S_{M_b}$; therefore also $M_a \supset M_b$, and hence there is a sequence $M_a, \ldots, M_b$ consisting only of $\supset$ flips w.r.t. $\mathcal{N}_M$.

For the inductive case assume that there exists a sequence $S_{M_a}, \ldots, S_{M_b}$ w.r.t. $\mathcal{N}_{S_M}$ with $k + 1 \geq 1$ cs_1 flips for some $S_{M_a} \in *M_a, S_{M_b} \in *M_b$. Consider the last cs_1 flip $S_{M_c} \rightsquigarrow S_{M_d}$ in the sequence, with $S_{M_c} \in *M_c, S_{M_d} \in *M_d$ for $M_c, M_b \subseteq M$. If the sequence $S_{M_a}, \ldots, S_{M_c}$ is trivial we have that $M_a = M_c$. Otherwise, by inductive hypothesis there is a sequence $M_a, \ldots, M_c$ w.r.t. $\mathcal{N}_M$ with k CI flips. Moreover, by Lemma 1 also $M_c \rightsquigarrow M_d$ is a CI flip w.r.t. $\mathcal{N}_M$. Finally, either $M_d = M_b$ (i.e. $S_{M_d}, \ldots, S_{M_b}$ is a trivial sequence) or $S_{M_d} \supset S_{M_b}$ (i.e. $S_{M_d}, \ldots, S_{M_b}$ involves $\supset$-flips) in which case $M_d \supset M_b$. In all cases we have a sequence $M_a, \ldots, M_c, M_d, \ldots, M_b$ w.r.t. $\mathcal{N}_M$ with $k + 1$ CI flips. □

Proposition 2. Let $\mathcal{N}_M$ be satisfiable and $M_a, M_b \subseteq M$. Then $M_a <_{\mathcal{N}_M} M_b$ iff $\widetilde{M_a} <_{\mathcal{N}_{S_M}} \widetilde{M_b}$.

Proof. If $M_a <_{\mathcal{N}_M} M_b$, then $M_a \hookrightarrow_{\mathcal{N}_M} M_b$. Hence, from Lemma 3 it follows that $\widetilde{M_a} <_{\mathcal{N}_{S_M}} \widetilde{M_b}$. Assume now $\widetilde{M_a} <_{\mathcal{N}_{S_M}} \widetilde{M_b}$ and, therefore, $\widetilde{M_a} \hookrightarrow_{\mathcal{N}_{S_M}} \widetilde{M_b}$. Note that then in fact $\widetilde{M_a} \rightharpoonup_{\mathcal{N}_{S_M}} \widetilde{M_b}$ (for any sequence $S' \hookrightarrow_{\mathcal{N}_{S_M}} S''$ consisting only in cs_2 flips it holds that $S_c, S_d \in *M_c$ for some $M_c \subseteq M$ and by assumption $M_a \neq M_b$.). Hence, from Lemma 3 it follows that $M_a \hookrightarrow_{\mathcal{N}_M} M_b$. □

Proposition 3. $\mathcal{N}_M$ is satisfiable iff $\mathcal{N}_{S_M}$ is satisfiable.

Proof. We prove that $\mathcal{N}_M$ is unsatisfiable iff $\mathcal{N}_{S_M}$ is unsatisfiable. Assume first that $\mathcal{N}_M$ is unsatisfiable. This means that there is an $M_a \subseteq M$ s.t. $M_a \hookrightarrow_{\mathcal{N}_M} M_a$. Hence, from Lemma 3 it follows that $\widetilde{M_a} \hookrightarrow_{\mathcal{N}_{S_M}} \widetilde{M_a}$, i.e. $\mathcal{N}_{S_M}$ is unsatisfiable. Assume now that $\mathcal{N}_{S_M}$ is unsatisfiable. Then there is a $S_{M_a} \in *M_a$ with $M_a \subseteq M$ s.t. $S_{M_a} \hookrightarrow_{\mathcal{N}_{S_M}} S_{M_a}$. In fact $S_{M_a} \rightharpoonup_{\mathcal{N}_{S_M}} S_{M_a}$ since $\mathcal{N}_{S_M}$ without the CI-statements has an acyclic dependency graph and is, therefore, satisfiable. Hence, from Lemma 3 it follows that $M_a \hookrightarrow_{\mathcal{N}_M} M_a$, i.e. $\mathcal{N}_M$ is unsatisfiable. □

B Translating confined reasoning about $C^{\aleph_0}I$-nets to reasoning about C^mI-nets

Let $\mathcal{N}$ be a $C^{\aleph_0}I$ net on O, $M \in \mathcal{M}_O$. We here sketch a translation of confined reasoning w.r.t. $\mathcal{N}$ and an $M \in \mathcal{M}_O$ to reasoning about a C^mI-net $\mathcal{N}'$ on M. Concretely, let $c = P^+ : P_1 \rhd P_2$ be a $C^{\aleph_0}I$ statement in $\mathcal{N}$. c is *meaningful w.r.t. M* if there is an $M' \in P^+$, s.t. $(M' \cup M_{P_1}) \subseteq M$, and $(M' \cup M_{P_2}) \subseteq M$. For our translation we rewrite each such $c \in \mathcal{N}$ into a C^mI-statement $c' \in \mathcal{N}'$ that is *equivalent to c for M*, i.e. the CI flips w.r.t. c' are exactly those in $\{(M' \cup M_{P_1}) \rightsquigarrow (M' \cup M_{P_2}) \mid (M' \cup M_{P_1}) \subseteq M, (M' \cup M_{P_2}) \subseteq M\}$. This means, the CI flips w.r.t. the resulting C^mI-net $\mathcal{N}' = \{c' \mid c \in \mathcal{N}\}$ are exactly those CI flips $M_a \rightsquigarrow M_b$ w.r.t. $\mathcal{N}$ s.t. $M_a, M_b \subseteq M$. As a consequence, $M_a \hookrightarrow_{\mathcal{N},M} M_b$ iff $M_a \hookrightarrow_{\mathcal{N}'} M_b$.

So assume $c = P^+ : P_1 \rhd P_2$ is a $C^{\aleph_0}I$- statement in $\mathcal{N}$ that is meaningful w.r.t. M. Then, since c is also satisfiable note that the precondition and comparison expressions can be written in the form $P^+ = \{o_i^+ \geq a_i^+\}_{1 \leq i \leq p} \cup \{o_j^- \leq a_j^-\}_{1 \leq j \leq q}$, $P_1 = \{o_k^1 + +a_k^1\}_{1 \leq k \leq r}$, $P_2 = \{o_l^2 + +a_l^2\}_{1 \leq l \leq s}$ where each $o \in O$ appears at most once in a sub-expression of the form $o_i^+ \geq a_i^+$ and at most once in a sub-expression of the form $o_j^- \leq a_j^-$ in the precondition. Let $O^* := \{o_i^+\}_{1 \leq i \leq p} \cup \{o_j^-\}_{1 \leq j \leq q} \cup \{o_k^1\}_{1 \leq k \leq r} \cup \{o_l^2\}_{1 \leq l \leq s}$. We re-label the objects in $O^* \subseteq O$ to $\{o_1, \ldots, o_m\}$ $(m = p + q + r + s)$. We now define for each $1 \leq h \leq m$, $A_h^x := a_i^x$ if there is a $t \in \{1, \ldots, y\}$ s.t. $o_h = o_t^x$, $A_h^x := 0$ otherwise for $x = +$, $x = 1$, $x = 2$ and $y = p$, $y = r$, $y = s$ respectively. Also, $A_h^- := a_j^-$ if there is a $j \in \{1, \ldots, q\}$ s.t. $o_h = o_j^-$, $A_h^- := m_M(o_h)$ otherwise. Finally, for each $1 \leq h \leq m$ we define $B_h^- := max\{I \mid A_h^+ \leq I \leq A_h^- \; and \; I + A_h^1 + A_h^2 \leq m_M(o_h)\}$. Then $c' \in \mathcal{N}'$ is the C^mI-statement $M^+, M^- : M_1 \rhd M_2$ where $M^+ := \{(o_h, A_h^+) \mid 1 \leq h \leq m, A_h^+ > 0\}$, $M^- := \{(o_h, X_{o_h}) \mid 1 \leq h \leq m, X_{o_h} > 0)\}$, $X_{o_h} := m_M(o_h) - B_h^- - A_h^1 - A_h^2$, $M^1 := \{(o_h, A_h^1) \mid 1 \leq h \leq m, A_h^1 > 0\}$, $M^2 := \{(o_h, A_h^2) \mid 1 \leq h \leq m, A_h^2 > 0\}$.