Computational Complexity in Constraint-based Combinatorial Auctions

Michael Thomas Egner Walter Hower

egnerm@gmail.com hower@hs-albsig.de

Abstract: This paper analyzes the dynamic programming construction of bundles within the framework of the Winner Determination Problem in Combinatorial Auctions, based on constraint processing. We discuss different approaches to its representation and highlight the corresponding complexity, employing suitable combinatorics from Discrete Mathematics. Our view may enlighten us about the exponential search space—and incidentally pointing to appropriate techniques to cope with this challenge.

1 Introduction

In the present paper, we focus on the Winner Determination Problem (WDP) in Combinatorial Auctions (CAs): "Given a set of bids in a combinatorial auction, find an allocation of items to bidders, including the possibility that the auctioneer retains some items, that maximizes the auctioneer's revenue."; cf. [CSS06], p. 8. No item is allocated more than once and every bidder receives at most one subset. In this paper, the WDP for the OR-bidding language is considered, i.e. all atomic bids by one bidder are connected with OR operators; the bidder is willing to obtain any number of disjoint atomic bids for the sum of their respective prices.

The WDP belongs to the $\mathcal{NP}$-complete problem class, i.e. it cannot be deterministically solved in polynomial time (as long as $\mathcal{NP} \neq \mathcal{P}$). This statement has encouraged us to check various forms of architectures to address the problem.

We mainly highlight the complexity to construct the bundles, to be grounded in a constraint-based interpretation; finally, we summarize the paper.

2 Representation

At first glance the representation of the CA bundles as a power-set lattice—as illustrated in Figure 1 for an auction with 4 items—looks very appealing. The bottom-up phase in [How98] in a global approach to the according constraint satisfaction problem (CSP) would produce possible solutions for the bundles on the next higher level. The feasible solutions for the top node (involving all variables) would then be solution candidates for the WDP (here a constraint optimization problem). For this representation, the decision

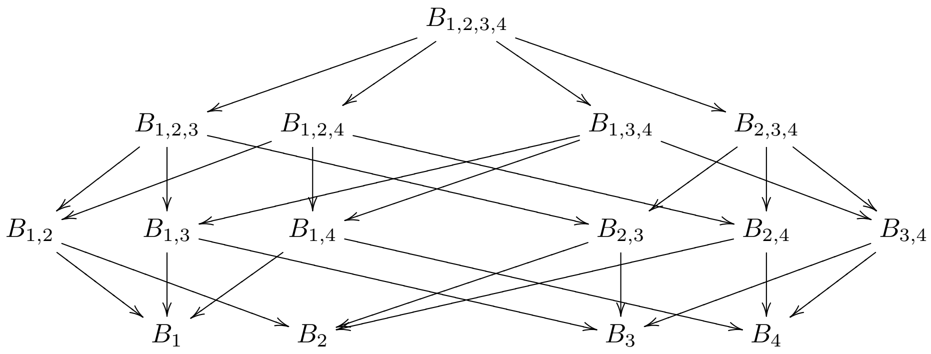

Figure 1: Power-set lattice

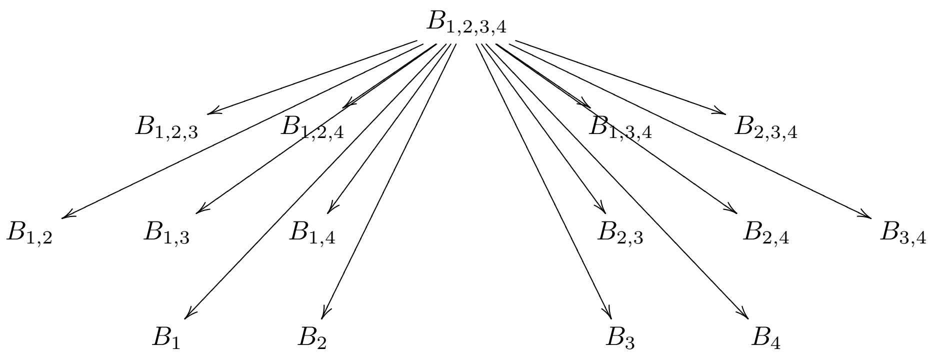

Figure 2: Lattice with joins for bundle $B_{1,2,3,4}$

has to be made whether the bidders or the items form the variables of such a CSP. Considering the first variant we would always have the (additional) task to maintain the feature of disjoint bundles. Hence, we have experimented taking the items as variables (the bidders serve as values), by which we (automatically) ensure an (appropriate) allocation.

The composition of bundles with more than two items cannot be implemented by only combining bundles on the next lower level. There may exist homogeneous bids for bundles which can't be split up: the values for the items in the bundle may all be identical—the same bidder may get assigned to each position in the bundle tuple; otherwise a bidder would receive bundles s/he didn't want at all. In contrast to the power-set lattice presented in Figure 1, not only bundles on the next lower level are part of the joining procedure. Figure 2 shows a lattice now with all bundles connected to bundle $B_{1,2,3,4}$ that are involved in the joins to construct all possibilities for $B_{1,2,3,4}$.

Additionally, bundles on lower levels can also be the result of joins. This leads to the lattice shown in Figure 3. Comparing the number of joins in Fig. 1 and Fig. 3, we can recognize that the really necessary number of joins is higher than naïvely expected.

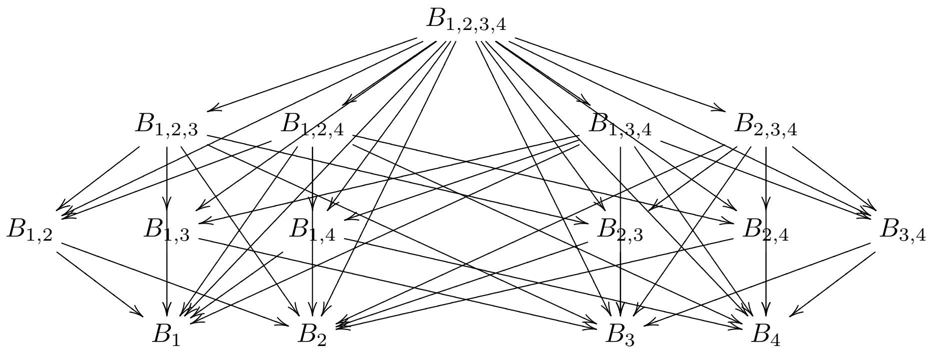

Figure 3: Lattice with all neccessary joins

Let us count the number of *p*roper non-empty subsets of a finite set of cardinalit*y* y:

$$p \quad := \quad \sum_{i:=1}^{y-1} \binom{y}{i} \quad = \quad \sum_{i:=0}^{y} \binom{y}{i} - \binom{y}{y} - \binom{y}{0} \quad = \quad 2^y - 1 - 1 \quad =$$

$$2 \cdot (2^{(y-1)} - 1) \quad . \tag{1}$$

Let us now count the number of all subset conne*x*ions downward of all set nodes at their cardinality *l*evel l:

$$x \quad := \quad \sum_{l:=1}^{m} \binom{m}{l} \cdot 2 \cdot (2^{(l-1)} - 1) \quad = \quad 2 \cdot \sum_{l:=1}^{m} \binom{m}{l} \cdot S_2(l,2) \quad =$$

$$2 \cdot \sum_{l:=2}^{m} \binom{m}{l} \cdot S_2(l,2) \quad , \tag{2}$$

m being the number of ite*m*s in the auction, $S_2(n,k)$ the *S*tirling No. of the 2nd kind—indicating by $S_2(l,2)$ the number of possibilities of a level-*l* set merging 2 sub-bundles.

Let us finally count[1] the actual number of *j*oins[2] having *i* *i*tems in the bundle(s) at cardinality level *i*:

$$j \quad = \quad x/2 \quad = \quad \sum_{i:=2}^{m} \binom{m}{i} \cdot S_2(i,2) \quad = \quad \sum_{i:=2}^{m} \binom{m}{i} \cdot (2^{(i-1)} - 1) \quad =$$

$$\left(\sum_{i:=2}^{m} \binom{m}{i} \cdot 2^{(i-1)} \right) - \left(\sum_{i:=2}^{m} \binom{m}{i} \cdot 1 \right) \quad =$$

[1] Thanks to Heinz Lüneburg for discussion
[2] correct—no “t” inside $\ddot\smile$

$$\left(\sum_{i:=0}^{m}\binom{m}{i}\cdot 2^{(i-1)}-\binom{m}{1}\cdot 2^{(1-1)}-\binom{m}{0}\cdot 2^{(0-1)}\right)-$$

$$\left(\sum_{i:=0}^{m}\binom{m}{i}-\binom{m}{1}-\binom{m}{0}\right) \quad =$$

$$\left(\left(\sum_{i:=0}^{m}\binom{m}{i}\cdot 1^{(m-i)}\cdot 2^{i}\right)/\,2-m\cdot 2^{0}-1\cdot 2^{(-1)}\right)-$$

$$\left(\sum_{i:=0}^{m}\binom{m}{i}\cdot 1^{(m-i)}\cdot 1^{i}-m-1\right) \quad =$$

$$((1+2)^{m}/2-m-1/2)-((1+1)^{m}-m-1) \quad =$$

$$(3^{m}-1)/2-m-2^{m}+m+1 \quad = \quad (3^{m}-1+2)/2\;-\;2^{(m+1)}/2 \quad =$$

$$(3^{m}+1-2^{(m+1)})\,/\,2 \quad . \tag{3}$$

This complete representation obviously exceeds the available resources.[3]

A different, more promising approach to the WDP is a heuristic one. Using techniques as presented in [SK06] could result in comparatively good—albeit possibly not optimal—solutions. The challenge in using such methods in connection with Combinatorial Auctions lies in modeling a sophisticated mutation procedure. Indiscriminately mutating entries may lead to infeasible collections that conflict with the idea that, in the end, bids may really get allocated. Here is one possibility to deal with this problem:

After (e.g., randomly) determining the number of positions to mutate and then the corresponding specific positions, the remaining partial allocations have to be checked for feasibility. If not feasible, this iteration must be skipped. In the other (positive) case (maybe gained by problem structure-preserving procedures), the next step would construct a valid allocation for the new position(s). Merging those two partial allocations inherently leads to a feasible overall allocation, and that allocation (the new or the former one) favoured by the heuristic—passing the acceptance criterion (deterministically/randomly, greedy/tolerance-based)—is used as the basis for the next iteration.

[3] Of course, in our small example, it's not tremendous:

$$\begin{aligned}
x &=_{[\texttt{Figure 3}]} 1\cdot(4+6+4)+4\cdot(3+3)+6\cdot 2\\
&= \quad 14 \quad + \quad 24 \quad + \quad 12 \quad = \quad 50\\
&=_{[\texttt{Equation 2}]} 2\cdot(\tbinom{4}{1}\cdot(2^{(1-1)}-1)+\tbinom{4}{2}\cdot(2^{(2-1)}-1)+\\
&\qquad \tbinom{4}{3}\cdot(2^{(3-1)}-1)+\tbinom{4}{4}\cdot(2^{(4-1)}-1))\\
&= 2\cdot(4\cdot(1-1)+6\cdot(2-1)+4\cdot(4-1)+1\cdot(8-1))\\
&= 2\cdot(\;0 \quad + \quad 6 \quad + \quad 12 \quad + \quad 7)=2\cdot 25\\
j &= x/2 = 25\cdot 2/2 = 25\\
&=_{[\texttt{Equation 3}]} (3^{4}+1-2^{(4+1)})/2=(81+1-32)/2=50/2
\end{aligned}$$

3 Remarks

The exponential outcome yields to a recommendation of a distributed system, by which each computer node reflects a bundle on an online combinatorial auction (CA) platform. From a bidder's point of view it might be advantageous to introduce several "bid bots" (*bid*ding ro*bots*)[4], yet belonging to the same physical bidder. Playing a role in several bundles hides the uniqueness of the actual bidder—a perhaps interesting feature reflecting asymmetric information. Furthermore, it is usually unknown whether bidders really compete or (at least temporarily) cooperate; such research has a tremendous impact—we may just recall the previous Nobel Prize in Economics "for having enhanced our understanding of conflict and cooperation through game-theory analysis", announced via [Nob05].

4 Résumé

The present paper illustrates the huge combinatorial search space in the auction setting by a thorough, though easily accessible, theoretical analysis based on Discrete Mathematics[5]—reflecting the still existing state of the art presented in [San02]. Only for a small-sized CA an exhaustive representation is possible. Whether local constraint propagation might help is still subject to additional investigation. An initial implementation in the frame of a modest research project is currently in progress and should provide further insight.

References

[CSS06] Peter Cramton, Yoav Shoham, and Richard Steinberg. *Combinatorial Auctions*. The MIT Press, Cambridge, Massachusetts, 2006.

[How98] Walter Hower. Revisiting global constraint satisfaction. *Information Processing Letters*, 66(1):41–48, 15 April 1998. Amsterdam, NL: Elsevier.

[How09] Walter Hower. *Diskrete Mathematik — Grundlage der Informatik*. Oldenbourg Wissenschaftsverlag, München, 2009. ISBN: 978-3-486-58627-5.

[Nob05] Nobelprize.org. The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005: Robert J. Aumann and Thomas C. Schelling. http://nobelprize.org/nobel_prizes/economics/laureates/2005/, 8 December 2005. Access: 1 July 2008.

[San02] Tuomas Sandholm. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence*, 135(1–2):1–54, February 2002. Amsterdam, NL: Elsevier.

[SK06] Johannes J. Schneider and Scott Kirkpatrick. *Stochastic Optimization*. Springer, Berlin / Heidelberg, 2006.

[4] incl. the artificial *b*idder b_0 getting assigned to each item retained (not purchased)

[5] you may consult [How09] ☺