# Server Models for Probabilistic Network Calculus

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**Abstract:** Network calculus is a deterministic queuing theory that has gained increasing attention in recent time. Founded on min-plus algebra it resorts to intuitive convolution formulae for efficient concatenation of servers and derivation of related performance bounds. Yet, the pessimistic worst-case analysis of deterministic network calculus gave rise to probabilistic counterparts that aim at utilizing the smoothing effects of statistical multiplexing by allowing for certain violation probabilities. Related theories are, however, significantly more complicated and still subject to research. To advance theory this paper evolves server models for probabilistic network calculus that are based on moment generating functions to efficiently utilize statistical multiplexing and the independence of flows.

## 1 Introduction

Network calculus [3, 7] is a min-plus system theory that facilitates the efficient derivation of performance guarantees for single servers and owing to a fundamental concatenation theorem also for networks. These service guarantees comprise deterministic delay, backlog, and output bounds. However, the conservative analysis of network calculus generally considers the worst-case and thus tends to overestimate resource requirements.

In current research this pessimistic view is relaxed by permitting bounds to be exceeded with certain usually small violation probabilities. Thereby the statistical gain obtained from multiplexing independent flows can be utilized efficiently to improve resource utilization. The issue of statistical multiplexing has gained significant attention, for example within the theory of effective bandwidth [3, 6], where moment generating functions of traffic arrivals are applied beneficially.

Using the Chernoff bound related traffic models have been adopted in the pioneering work on stochastic $(\sigma, \rho)$-calculus [2] which is continued in [3] and introduced to the framework of network calculus in [1] where the notable concept of effective envelopes is devised. The relation to the theory of effective bandwidth is elaborated in [8]. Recently, a general network calculus with moment generating functions was derived in [5], where this work evolves corresponding server models.

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## 2 Preliminaries

In the context of network calculus flows are described by cumulative arrival functions $F(t)$, where $F(t)$ is the amount of data seen in an interval $[0,t]$ with $t \in \mathbb{N} = \{0, 1, 2, \dots\}$. Thus, $F(0) = 0$ and $F(t)$ is increasing, that is $F(t) \geq F(s)$ for all $t \geq s$ and $s \geq 0$.

In contrast to known deterministic network calculus [3, 7] we use a definition of arrival and service curves at a given local time instance $u$ as proposed in [5].

**Definition 1 (Local Arrival and Service Curves).** Consider an increasing stochastic sequence $F(t)$ which describes the cumulative arrival function of a traffic flow. Then $A_u(t)$ is a local arrival curve at time $u$ if for all $t \geq 0$ and any $u \geq t$

$$A_u(t) \geq F(u) - F(u-t).$$

Assume the sequence $F(t)$ is input to a network element and $F'(t)$ is the respective output. Then $B_u(t)$ is a local service curve at time $u$ if for any $u \geq t$.

$$F'(u) \geq \inf_{t \in [0,u]} [F(u-t) + B_u(t)].$$

Note that the definition of local service curve corresponds to the concept of dynamic server with time varying capacity in [3]. However, in this work the dependence on the local time instance $u$ is eliminated by assuming stationarity such that the index $u$ can be dropped [5].

The probabilistic network calculus in [5] builds on moment generating functions of traffic arrivals and offered service as defined below.

**Definition 2 (Moment Generating Function).** The moment generating function of a stochastic sequence $A(t)$ is defined for any $\theta$ as

$$M_A(\theta, t) = \mathsf{E} e^{\theta A(t)} = \sum_{a=-\infty}^{\infty} e^{\theta a} \mathsf{P}\{a = A(t)\}.$$

We define the conjugate moment generating function of a stochastic sequence $B(t)$ for any $\theta$ as $\overline{M}_B(\theta, t) = M_B(-\theta, t)$.

**Corollary 3 (Addition and Multiplication of Constants).** *For addition and multiplication of constants $c_1$ respective $c_2$ it follows for all $\theta$ that*

$$M_{c_1+c_2A}(\theta, t) = e^{c_1\theta} M_A(c_2\theta, t),$$
$$\overline{M}_{c_1+c_2B}(\theta, t) = e^{-c_1\theta} \overline{M}_B(c_2\theta, t).$$

**Corollary 4 (Addition of Independent Stochastic Sequences).** *It follows for the sum respective difference of independent stochastic sequences $A(t)$ and $B(t)$ for all $\theta$ that*

$$M_{A+B}(\theta, t) = M_A(\theta, t) M_B(\theta, t),$$
$$\overline{M}_{A+B}(\theta, t) = \overline{M}_A(\theta, t) \overline{M}_B(\theta, t),$$
$$M_{A-B}(\theta, t) = M_A(\theta, t) \overline{M}_B(\theta, t),$$
$$\overline{M}_{A-B}(\theta, t) = \overline{M}_A(\theta, t) M_B(\theta, t).$$

**Corollary 5 (Infimum and Supremum of Stochastic Sequences).** *For the infimum respective supremum of two stochastic sequences $A(t)$ and $B(t)$ it follows for $\theta \geq 0$ that*

$$M_{\inf[A,B]}(\theta, t) \leq \inf[M_A(\theta, t), M_B(\theta, t)],$$
$$\overline{M}_{\sup[A,B]}(\theta, t) \leq \inf[\overline{M}_A(\theta, t), \overline{M}_B(\theta, t)].$$

# 3 Probabilistic Server Models

Starting from a general scheduling discipline we derive probabilistic models for priority scheduling (PS), generalized processor sharing (GPS) and first-in first-out (FIFO) scheduling for $n$ concurrent flows shown in Fig. 1. We use the convention that $i, j, k \in [0, n]$.

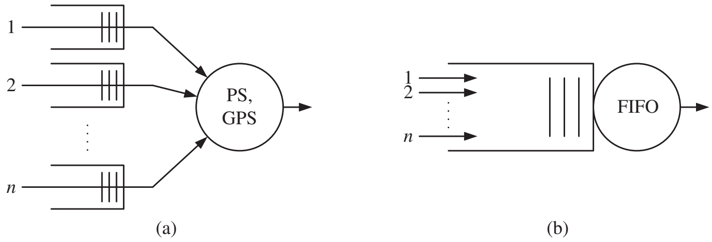


Figure 1: Priority scheduling, generalized processor sharing, and first-in first-out server models.

**General Scheduling Discipline** If the arbitration between flows is unknown the following result, which is conservative for most scheduling disciplines, can be derived [7].

**Lemma 6 (General Scheduling Discipline).** *Consider $n$ flows that traverse a network element with local service curve $B_u(t)$. Assume that the flows are upper constrained by local arrival curves $A_{u,j}(t)$. Then, a local service curve offered to flow $i$ is given by*

$$B_{u,i}(t) = \sup\left[0, B_u(t) - \sum_{j \neq i} A_{u,j}(t)\right].$$

*Proof.* Let $F_j(t)$ and $F'_j(t)$ be the cumulative arrival functions of the flows as they are input to respective output from the network element. With the definition of local service curves, the rule for multiplexing, and since $F_j(t) \geq F'_j(t)$ for all $j$ and all $t \geq 0$ we have:

$$\exists t \in [0, u] : \sum_j F'_j(u) \geq \sum_j F_j(u - t) + B_u(t)$$
$$\Rightarrow \exists t \in [0, u] : F'_i(u) \geq F_i(u - t) + B_u(t) - \sum_{j \neq i} (F_j(u) - F_j(u - t))$$

With Def. 1 the proof is complete. The lower bound of zero is fulfilled trivially. □

**Corollary 7 (General Scheduling Discipline).** *Consider the scenario in Lem. 6. Assume the offered service is stationary and has conjugate moment generating function* $\overline{M}_B(\theta, t)$ *and the traffic arrivals are stationary and have moment generating functions* $M_{A_j}(\theta, t)$*. Under the assumption of independence the conjugate moment generating function of the service that is offered to flow* $i$ *is upper bounded for* $\theta \geq 0$ *according to*

$$\overline{M}_{B_i}(\theta, t) \leq \inf\left[1, \overline{M}_B(\theta, t) \prod_{j \neq i} M_{A_j}(\theta, t)\right].$$

**Priority Scheduling** A priority scheduler serves queues which are ordered by decreasing priority, such that a higher label represents a lower priority. Each time the scheduler chooses the earliest arrival from the queue with the highest priority that is non-empty. Under the discrete time model we assume that preemption can take place at any time instance. The impacts of variable length packets and non-preemptive scheduling are addressed in [3, 7]. For the service offered to flow $i$ it follows immediately from the prioritization that

$$B_{u,i}(t) = \sup\left[0, B_u(t) - \sum_{j=0}^{i-1} A_{u,j}(t)\right],$$
$$\overline{M}_{B_i}(\theta, t) \leq \inf\left[1, \overline{M}_B(\theta, t) \prod_{j=0}^{i-1} M_{A_j}(\theta, t)\right].$$

**Generalized Processor Sharing** In case of generalized processor sharing [9] a weight $\phi_i$ is assigned to each of the $n$ traffic classes, where traffic class $i$ receives a share of $\phi_i / \sum_k \phi_k$ of the available service if all of the $n$ queues are backlogged. If any class $j$ uses less than the assigned service, the remaining service is distributed among the other classes according to the respective weights. The theoretical concept of generalized processor sharing relies on a fluid flow model, whereas extensions for packet-by-packet scheduling are provided for example in [3, 7]. The following model provides a conservative approximation which is best for the homogeneous case where the terms $\overline{M}_B(\phi_j / \sum_k \phi_k \theta, t) M_{A_j}(\theta, t)$ do not or only marginally depend on $j$.

$$B_{u,i}(t) \geq \sup\left[\tfrac{\phi_i}{\sum_k \phi_k} B_u(t), B_u(t) - \sum_{j \neq i} A_{u,j}(t)\right],$$
$$\overline{M}_{B_i}(\theta, t) \leq \inf\left[\overline{M}_B\left(\tfrac{\phi_i}{\sum_k \phi_k} \theta, t\right), \overline{M}_B(\theta, t) \prod_{j \neq i} M_{A_j}(\theta, t)\right].$$

**FIFO Aggregate Scheduling** A parameterized family of service curves with parameter $\tau \geq 0$ is presented in [4] for flows that are served as an aggregate in first-in first-out order. If the order is unknown $\tau = 0$ applies and the general scheduling discipline is recovered. Using the definitions of local arrival and service curves the following result can be obtained in the same line as the derivation for the deterministic case in [7], where the indicator function $1_{[\dots]}$ is one if the argument is true and zero otherwise.

$$B_{u,i}(t) = \sup\left[0, B_u(t) - \sum_{j\neq i} A_{u-\tau,j}(t-\tau)\right] 1_{[t>\tau]},$$

$$\overline{M}_{B_i}(\theta,t) \leq \inf\left[1, \overline{M}_B(\theta,t) \prod_{j\neq i} M_{A_j}(\theta, t-\tau)\right] 1_{[t>\tau]} + 1_{[t\leq\tau]}.$$

## 4 Conclusions

A variety of efficient models for statistical multiplexing of independent flows are known, for example from the theory of effective bandwidth. In this paper we derived per-flow service curves which constitute the basis for a system theoretic view on networks of queues that features the analysis of flows after de-multiplexing, an issue that is not well understood, yet. While traffic models that are based on moment generating functions are known for a variety of types of flows, we advance theory by providing corresponding models for a number of widely-used scheduling disciplines which enables the probabilistic analysis of network elements beyond simple, for example constant rate, servers.

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