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Formal Algorithmic Elimination for PDEs
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Zusammenfassung:

This thesis approaches partial differential equations (PDEs) from the viewpoint of algebra and contributes algorithmic methods which allow to investigate effectively the relationship of systems of PDEs and their sets of solutions. Employing formal techniques, the focus is on polynomial differential equations and their analytic solutions.

We borrow quite a few concepts from algebraic geometry. Whenever a set of points is given by a polynomial or rational parametrization, an elimination of the parameters from the equations which express the coordinates of the points yields equations that are satisfied by the coordinates of every point of the set. If there exists an implicit description of this set as solution set of a system of polynomial equations, then elimination constructs such a description.

This thesis develops algorithmic methods which accomplish the analogous elimination task for systems of polynomial partial differential equations and their (complex) analytic solutions. It builds on work by C. Riquier, M. Janet, J. M. Thomas, J. F. Ritt, E. R. Kolchin, and others, who laid the foundation of differential algebra.

A given multivariate polynomial, whose coefficients are analytic functions, is interpreted as a parametrization of a set of analytic functions, i.e., every element of this set arises from substitution of appropriate analytic functions for the indeterminates of the polynomial. Moreover, the substitution of functions for the indeterminates also involves the composition with prescribed analytic functions. If the polynomial is linear, then the resulting set is a vector space over the field of constants. In general, however, the parametrized set is rarely closed under addition.

As a simple example we mention that the analytic functions of the form $F(x-t)+G(x+t)$ admit an implicit description as solution set of the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, a well-known fact when considered in the reverse direction.

An implicit description of an algebraic variety admits a straightforward check whether or not a given point belongs to the variety, namely by verifying if it is a solution of the defining equations. Similarly, if a set of analytic functions is the solution set of a system of differential equations, membership to this set reduces to the check whether or not a given analytic function is annihilated by the corresponding differential operators.

Depending on the representation of a function at hand, it may not be obvious at all whether the function has another representation of a special form, e.g., as sum of functions depending on a smaller number of arguments, etc. A prominent example, which is only loosely related to the contents of this thesis, is V. I. Arnold's solution of Hilbert's 13th problem showing that every continuous function of several variables can be represented as a composition of finitely many continuous functions of two variables. We restrict our attention to decomposition problems for analytic functions which may be answered by constructing a system of partial differential equations and inequations, whose set of solutions coincides with the set of decomposable functions.

The algorithmic methods developed in this thesis allow to improve symbolic solving of PDEs. On the one hand, membership of a solution to a family of solutions, which is implicitly described by PDEs, is decidable, so that questions regarding completeness of the family of solutions can be addressed. On the other hand, adding a PDE system characterizing analytic functions of a special form to a PDE system to be investigated, may allow to extract explicit solutions of the prescribed type. This approach generalizes the well-known method of separation of variables. A small family of explicit solutions for the Navier-Stokes equations is computed.