who attended a three-hour seminar on the effort. "Experts in finite group theory and design theory should leap at the opportunity to make a contribution". Even if someone manages to prove one of the conjectures – thereby demonstrating that $\omega = 2$ – the wreath product approach is unlikely to be applicable to the large matrix problems that arise in practice. Although the wreath product algorithms perform better than Strassen asymptotically, Umans says, the input matrices must be astronomically large for the difference in time to be apparent. Still, the discovery of other classes of groups that satisfy the two conditions would provide new toolkits for developing fast matrix multiplication algorithms, possibly even practical ones. "One of the things that has kept us enthusiastic about this method is that many things have worked out in a beautiful way," Umans says. "This makes it seem like the right approach."

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Mellin-Barnes Integrals in Elementary Particle Physics

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Michal Czakon stammt aus Katowice, Polen, wo er Physik studiert und promoviert hat. Als Humboldt-Stipendiat ging er 2001 an die Universität Karlsruhe und wurde 2003 Postdoc am DESY Zeuthen. Er bekam 2004 für seine Leistungen zur Automatisierung von Multiloop-Rechnungen in der Teilchenphysik den Sofja-Kovalevskaya-Preis der Humboldt-Stiftung verliehen. Neben dem eigentlichen Preisgeld sieht dieser Preis die Ausstattung einer Nachwuchsgruppe um den Preisträger vor, die Michal Czakon an der Universität Würzburg aufgebaut hat.

Abstract

Many observable quantities in Elementary Particle Physics are predicted by means of a series expansion (perturbation theory) in a small coupling constant with the expansion coefficients given by complicated multidimensional integrals. We describe a MATHEMATICA *package [1] implementing techniques that allow for efficient evaluation of such integrals through Mellin-Barnes representations. The usefulness of the methods extends actually to any problems involving integrals with isolated singularities.*

The problem to be solved can be specified as follows. Let there be an integral, most often multidimensional, depending, amongst others, on a set of parameters $\epsilon, \eta, \ldots$ (typically just one), in which it has isolated singularities on the real axis. We wish to find a Laurent expansion in these parameters.

In many cases and in particular in those occurring in perturbation theory, the integral can be transformed into the following form

$$\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \ldots \int_{-i\infty}^{i\infty} \Pi_i dz_i \; f(z_1, \ldots, z_n, s_1, \ldots \ldots, s_p, a_1, \ldots, a_q, \epsilon) \frac{\Pi_j \Gamma(A_j + V_j + c_j \epsilon)}{\Pi_k \Gamma(B_k + W_k + d_k \epsilon)}, \quad (1)$$

where the dependence on only one of the interesting parameters, ϵ, has been kept explicit; the s_i may be complex; a_i are assumed to be rational; A_i, B_i are linear combinations of the a_i; V_i, W_i are linear combinations of the z_i; and c_i, d_i are some numbers. The function f is analytic, in practice a product of powers of the s_i, with exponents being linear combinations of the remaining parameters. Γ is the Euler Gamma function and its derivatives are also allowed.

Equation (1) is called a Mellin-Barnes (MB) integral, because it is usually obtained for integrals of products of polynomials raised to some powers with the help of the Mellin-Barnes representation

$$\frac{1}{(A+B)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz \frac{A^z}{B^{\nu+z}} \Gamma(-z) \Gamma(\nu + z), \quad (2)$$

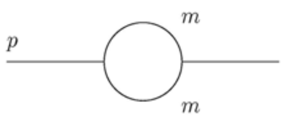
which is then integrated with a generalization of the Euler formula

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \qquad (3)$$

It is important that the contour in (2) separates the poles of the Γ functions with $+z$ and $-z$. This usually puts constraints on the contours in (1).

The idea is to start from some value of ϵ (or any other parameter) lying in a region of the complex plane, where the integral is analytic, and perform a continuation to the vicinity of the interesting point, e.g. the origin. Changing the value of ϵ will lead to crossing of the poles of the Γ-functions by the contours. The residues so obtained may then contain explicitly singular Γ-functions, whereas the integrals themselves are always regular. In this way, the Laurent series is explicitly constructed by expanding under the integral sign, the singularities coming from the prefactors. Notice that the starting region is determined by such contours and value of ϵ that all the arguments of the Γ-functions be positive.

Let us consider a very simple example, connected to the following perturbation theory diagram.



The associated integral can be written as

$$\Gamma(\epsilon)\int_0^1 dx\, (-sx(1-x)+m^2)^{-\epsilon}, \qquad (4)$$

and we are interested in the behavior of the integral as $\epsilon \to 0$. As it stands above the integral is already in a form that can be directly expanded. In order to show the main features of the software, we will derive an MB representation that is more complicated. After loading the `MB.m` package the suitable representation is obtained with

```
In[1]:= int = b0[s, 1+z1, 1+z2]*ms^z1*
        ms^z2*Gamma[-z1]*Gamma[1+z1]*
        Gamma[-z2]*Gamma[1+z2] /. z1
        -> z1-z2

Out[1]:= (m1s^z1*(-s)^(-ep - z1)*
         Gamma[ep + z1]*Gamma[1 - ep -
         z2]*Gamma[-z2]*Gamma[-z1 +
         z2]*Gamma[1 - ep - z1 + z2])/
         Gamma[2 - 2*ep - z1]
```

The user must now determine the contours, or more precisely, the real parts of the contours, since the package always assumes that the contours are parallel to the imaginary axis.

```
In[2]:= rules = MBoptimizedRules [int,
        ep -> 0, {}, {ep}]

MBrules::norules: no rules could be
found to regulate this integral

MBrules::norules: no rules could be
found to regulate this integral

Out[2]:= {{ep -> 7/8}, {z1 -> -3/4,
         z2 -> -1/2}}
```

The two warning messages have been generated during the determination of the contours, and since some real parts have been found, they are harmless and can be ignored.

The user can now perform the analytic continuation:

```
In[3]:= cont = MBcontinue[int,
        ep -> 0, rules]

Level 1

Taking +residue in z1 = -ep

Level 2

Integral {1}

Taking +residue in z2 = -ep

Level 3

Integral {1, 1}

3 integral(s) found

Out[3]:= {{MBint[(Gamma[1 - ep]*
          Gamma[ep])/(m1s^ep*
          Gamma[2 - ep]),{{ep -> 0},
          {}}]},  MBint[(Gamma[1 -
          ep - z2]*Gamma[-z2]*
          Gamma[1 + z2]* Gamma[ep +
          z2])/(m1s^ep*Gamma[2 - ep]),
          {{ep -> 0}, {z2 -> -1/2}}]},
          MBint[(m1s^z1*(-s)^(-ep -
          z1)*Gamma[ep + z1]*Gamma
          [1 - ep - z2]*Gamma[-z2]*
          Gamma[-z1 + z2]* Gamma[1 -
          ep - z1 + z2])/Gamma[2 -
          2*ep - z1],{{ep -> 0},
          {z1 -> -3/4, z2 -> -1/2}}]}
```

At this stage, the user can, for example, expand the integrals to determine the poles:

```
In[4]:= div = MBexpand[cont,
        Exp[ep EulerGamma], {ep, 0, -1}]

Out[4]:= {{{MBint[ep^(-1), {{ep -> 0},
         {}}]}}}
```

The second argument of MBexpand is a normalization factor. It is easy to see that this corresponds exactly to the singularity of the integral (4). The integral can now be integrated numerically with

```
In[5]:= num = MBexpand[cont,
        Exp[ep EulerGamma],{ep, 0, 0}];

In[6]:= MBintegrate[num, {s -> -1,
        ms -> 1}]// InputForm

Shifting contours...

Performing 1 1-dimensional
integrations...

1 Higher-dimensional integrals

Preparing MBpart1ep0 (dim 2)

Running MBpart1ep0

{-0.1520450250804735 + ep^(-1),
{0.0000149000407288098, 0}}
```

The result contains the numeric value of the integral and an estimate of the error. The actual integration is performed with FORTRAN programs and the CERNlib [2] and CUBA [3] libraries.

Up to now, the software has been used from moderately complicated integrals as

$$
\begin{aligned}
&\frac{1}{(2\pi i)^4}\frac{1}{\Gamma(-2\epsilon)}\int\limits_{-i\infty}^{i\infty}\int\limits_{-i\infty}^{i\infty}\int\limits_{-i\infty}^{i\infty}\int\limits_{-i\infty}^{i\infty} dz_1\, dz_2\, dz_5\, dz_6\, (-s)^{-2-2\epsilon-z_5-z_6}\left(\frac{t}{s}\right)^{z_1} \\
\times\;& \frac{\Gamma(-z_1)\Gamma(1+z_1)\Gamma(-1-2\epsilon-z_2)\Gamma(1+z_2)\Gamma(-3-4\epsilon-2z_1-z_2-2z_5)}{\Gamma(-1-2\epsilon-z_2-2z_5)\Gamma(-3\epsilon-z_5)\Gamma(-3-4\epsilon-2z_1-z_2-2z_5-2z_6)} \\
\times\;& \Gamma(-1-\epsilon-z_5)\Gamma(-\epsilon-z_2-z_5)\Gamma(-z_5)\Gamma(2+\epsilon+z_1+z_2+z_5) \\
\times\;& \Gamma(-1-2\epsilon-z_1-z_5-z_6)\Gamma(-2-2\epsilon-z_1-z_2-z_5-z_6)\Gamma(-z_6) \\
\times\;& \Gamma(2+2\epsilon+z_1+z_5+z_6),
\end{aligned}
$$

up to 14-fold integrals leading to tens of thousands of residues. It has proved to be very useful for numeric checks as well as for analytic evaluation. It is clear that due to the generality of the problem it could be applied to other branches of physics and mathematics. A detailed description can be found in [4]. If the reader would like to learn more about MB integrals in perturbation theory, he should consult the book [5].

Literatur

[1] MB.m package obtainable from http://theorie.physik.uni-wuerzburg.de/~mczakon

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[3] T. Hahn, Comput. Phys. Commun. **168** (2005) 78.

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