Tropical Geometry in SINGULAR

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Introduction

Tropical geometry studies balanced polyhedral complexes which arise in numerous areas of mathematics and beyond. In SINGULAR [4] we are naturally interested in its application in algebraic geometry.

Given a polynomial ideal $I \trianglelefteq K[\underline{x}] = K[x_1, \dots, x_n]$ over a field K with possibly trivial valuation $\nu : K \to \mathbb{R}$, we would like to determine its tropical variety

$$\mathrm{Trop}_\nu(I) := \{w \in \mathbb{R}^n \mid \mathrm{in}_{\nu,w}(I) \text{ monomial free}\},$$

as it inherently carries information about the affine algebraic variety $X := V(I) \subseteq k^n$ cut out by I (see [11] for details). Tropical geometers sometimes refer to them as combinatorial shadows of their algebraic counterparts.

For example, enumerative geometers have been studying tropical varieties to count algebraic curves with carefully chosen characteristics (see Figure 1). While counting, it is important to recognize if multiple objects are casted to the same shadow and, if needs be, determine that number of objects. The theorem that proved this to be possible is referred to as *Mikhalkin's Correspondence Theorem*.

While studying tropical varieties, sometimes it is helpful to compute concrete examples. Up until now, the only software that has been able to do so was GFAN [6] by A. N. Jensen, whose algorithms were developed in a collaboration with Bogart, Speyer, Sturmfels and Thomas [1]. However GFAN is restricted to the rational numbers $K = \mathbb{Q}$ and the valuation $\nu = 0$ being trivial, i.e. p-adic valuations ν_p are excluded. Nonetheless, it is also possible to compute over the field of rational Laurent series $K = \mathbb{Q}((t))$ with its natural valuation thanks to the trick of homogenization and dehomogenization.

The difficulty of non-trivial valuations comes from the fact that the classical Gröbner basis theory does not take any valuations on its ground field into account, as it relies solely on a chosen ordering on the monomials. This makes it unsuited for the problem at hand.

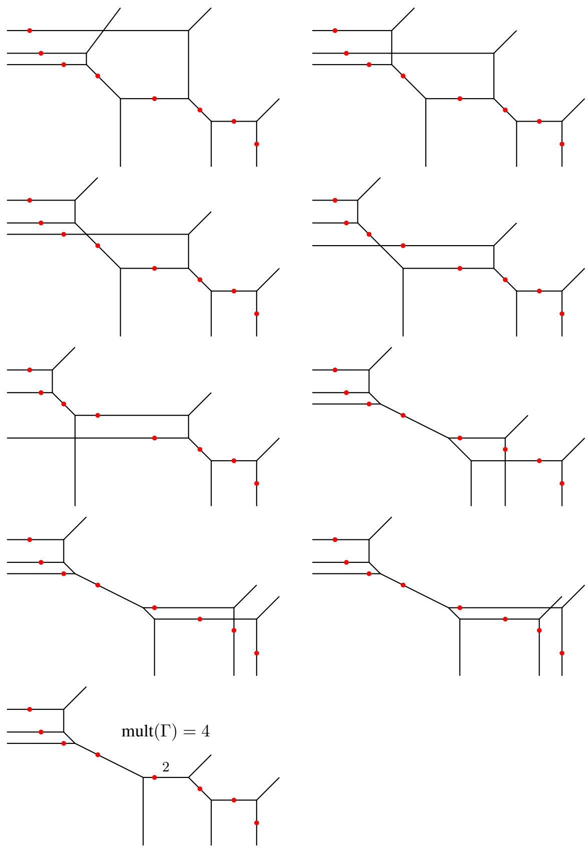

Figure 1: *tropical curves of degree 3 and genus 0 through 8 points in general position*

Progress to date

Ever since version 3-1-6, SINGULAR has been supporting convex geometry thanks to two interfaces [7, 9] to GFANLIB [6] and POLYMAKE [5] respectively. And, in version 4-0-2, we have successfully implemented algorithms for computing tropical varieties over $\mathbb{Q}$ with respect to both trivial and p-adic valuations.

The algorithms for the trivial valuation were taken from the existing work [1], while for p-adic valuations we have developed new techniques that allow us to fall back to the trivial valuation [10]. We are effectively tracing any tropical variety over $\mathbb{Q}$ under a p-adic valuation to a tropical variety over $\mathbb{Z}$ under the trivial valuation. However, the new ideals over $\mathbb{Z}$ are of more general form than the old ideals over $\mathbb{Q}$ for which all existing algorithms were designed, so that new computational approaches had to be developed. During this, we are heavily relying on SINGULAR's native standard basis engine for coefficient rings in arbitrary orderings.

In a way, this new technique can be seen as a generalization of the homogenization and dehomogenization strategy to compute over $\mathbb{Q}((t))$:

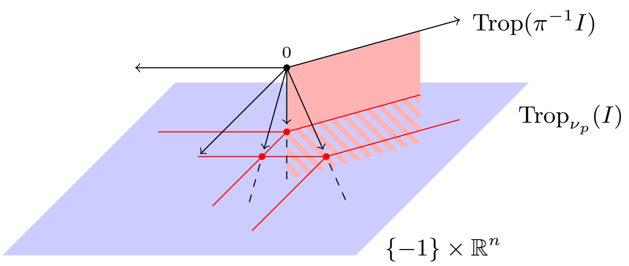

Figure 2: *tropical varieties with p-adic valuation over $\mathbb{Q}$ and trivial valuation over $\mathbb{Z}$ respectively*

During implementation, we paid special attention to unify our new algorithms for tropical varieties over $\mathbb{Z}$ and the existing algorithms for tropical varieties over $\mathbb{Q}$ in a common framework. All algorithms were implemented as part of the GFANLIB interface and are publicly available as part of the official SINGULAR distribution.

To compute tropical varieties, load `gfanlib.so` and use the command `tropicalVariety`. The command takes an ideal as first argument and has an optional second argument depending on which valuation you want to compute with:

```
                     SINGULAR                         / Version
 A CAS for Polynomial Computations                   /   4.0.2
                                                   0<
 Decker, Greuel, Pfister, Schoenemann              \   March
 FB Mathematik der TU Kaiserslautern                \   2015
> LIB "gfanlib.so";
> ring r = 0,(x,y,z,w),dp;
> ideal I = x+2y-3z, 3y-4z+5w;
> tropicalVariety(I,number(2)); // 2-adic val.
RAYS
-2 -1 1 -1 1     # 0
-1 1 -1 1 -1     # 1
0 -3 1 1 1       # 2
0 1 -3 1 1       # 3
0 1 1 -3 1       # 4
0 1 1 1 -3       # 5
LINEALITY_SPACE
0 -1 -1 -1 -1    # 0
MAXIMAL_CONES
{0 1}     # Dimension 3
{0 2}
{0 4}
{1 3}
{1 5}
> tropicalVariety(I,number(3)); // 3-adic val.
> tropicalVariety(I,number(5)); // 5-adic val.
> tropicalVariety(I,number(7)); // 7-adic val.
> tropicalVariety(I,number(11)); // 11-adic val.
> tropicalVariety(I); // trivial valuation
```

Figure 3: *computing tropical varieties of the same ideal with respect to multiple valuations on $\mathbb{Q}$*

As sketched in Figure 2, for p-adic valuations the output is a polyhedral fan whose intersection with the affine hyperplane on which the first coordinate is -1 yields the wanted tropical variety. This is akin to how POLYMAKE represents polyhedra and polyhedral complexes. The tropical varieties are combinatorially of the form:

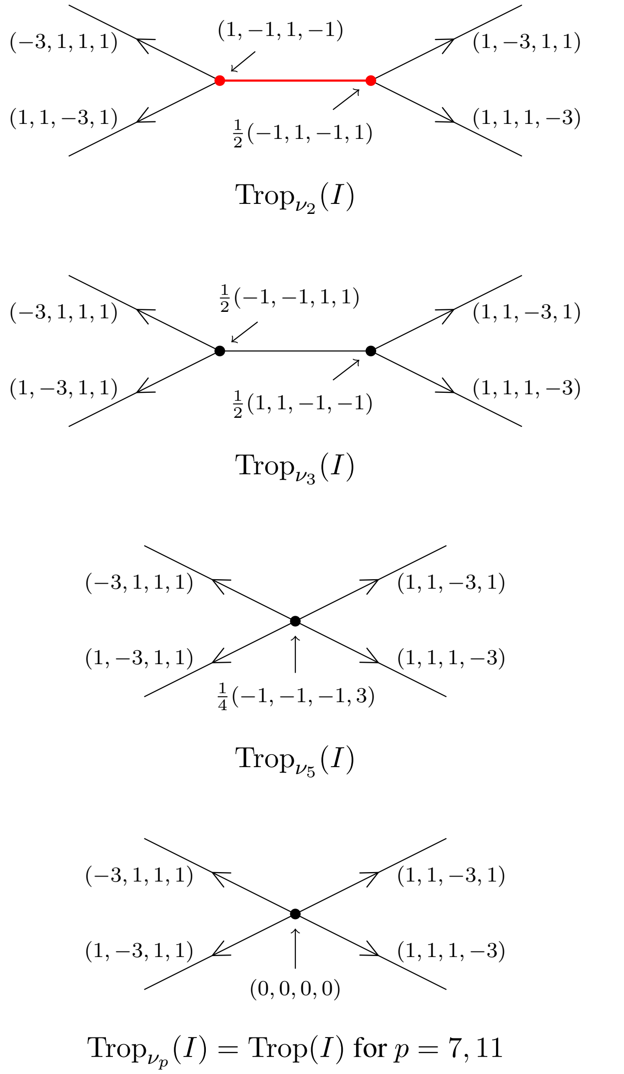

Figure 4: *tropical varieties of the same ideal with respect to multiple valuations on $\mathbb{Q}$*

The intersection of the affine hyperplane with the two highlighted rays in Figure 3 yield two distinct vertices in $\mathrm{Trop}_{\nu_2}(I)$ of Figure 4, whereas the intersection with the highlighted maximal cone yields the bounded edge connecting them. The remaining rays represent points at infinity, which is why the remaining maximal cones represent unbounded edges. The tropical variety has a lineality space generated by the weight vector $(1,1,1,1)$, which is due to the homogeneity of our input ideal.

Note how the tropical varieties dance around for the 2-, 3- and 5-adic valuation before settling to what is obtained with respect to the trivial valuation. This suggests that 2, 3 and 5 are so called bad primes for the modular techniques involving our ideal [2], which is no surprise as they appear as coefficients of our generators.

Current and Future work

One major bottleneck in our computation are standard bases computations over $\mathbb{Z}$ in arbitrary orderings. Thanks to the theory of Gröbner walks, it is only necessary once at the very beginning. There is currently a group of SINGULAR developers actively working on it, including Christian Eder, Anne Frübis-Krüger, Gerhard Pfister and Adrian Popescu, and any improvement will greatly benefit our performance for p-adic valuations. However, it may also be worthwhile to consider algorithms tailored to our ideals which exploit some of the common structure that they share [8].

Moreover, starting with the next version SINGULAR will support modified standard bases algorithms. One promising candidate for tropical computations is the so called *saturating standard bases algorithm*, in which each new basis element is checked for divisibility by the variables. Because we assume our ideal to be saturated with respect to all variables to begin with, we may use it indiscriminately without altering our ideal. This technique was originally applied in the monomial tests during the study of GIT-fans to great success [3], and we hope that it will do equally well in the massive amount of monomial tests in our tropical algorithms.

Feature-wise, the biggest priorities are the computation of multiplicities and the ability to exploit symmetries in our computations.

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