Investigating Terao's freeness conjecture with computer algebra

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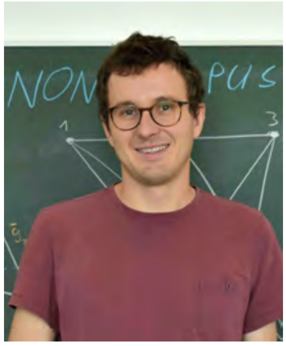

Introduction

In computational mathematics one often encounters the problem of scanning (finite but) large sets of certain objects. Here are two typical scenarios:

- Searching for a counter-example of an open conjecture among these objects.
- Building a database of such objects with some of their invariants.

A database is particularly useful when the questions asked are relational, i.e., involve more than one object. Recognized patterns and questions which a database answers affirmatively may lead to working hypotheses or even proofs by inspection.

This article describes an instance of this principle in an on-going research project in which we investigate hyperplane arrangements through a database [BK19b, BBJ19, BK21]. This is in part joint work with Reimer Behrends, Christopher Jefferson, and Martin Leuner.

The theory of hyperplane arrangements naturally lies at the crossroads of algebra, combinatorics, algebraic geometry, representation theory and topology. We are specifically interested in the interplay between the combinatorial and algebraic properties of an arrangement motivated by Terao's freeness conjecture, one of the central open conjectures in this area.

Hyperplane arrangements

We briefly give the relevant definitions together with an illustrating example in this section. We refer to the excellent textbook by Orlik and Terao for an in-depth introduction to the theory of hyperplane arrangements [OT92].

Definition 1 An *arrangement of hyperplanes* $\mathcal{A}$ is a finite set of hyperplanes in a vector space V over some field k. We assume that all hyperplanes contain the origin of V. We say that $\mathcal{A}$ is of rank ℓ if V is an ℓ-dimensional vector space.

Example 2 The rank 3 *braid arrangement* $\mathcal{A}_3$ is an arrangement in $\mathbb{R}^3$ defined via the linear equations

$$x = 0, y = 0, z = 0, x - y = 0, x - z = 0, y - z = 0.$$

Figure 1 shows a projectivized picture of $\mathcal{A}_3$, that is a slice of the arrangement with the hyperplane $z = 1$. The hyperplane $z = 0$ is shown as a circular line at infinity.

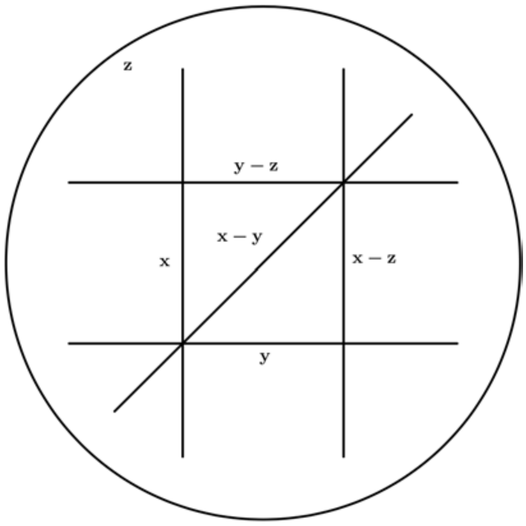

Figure 1: *The braid arrangement* $\mathcal{A}_3$.

Intersection lattice of an arrangement

Combinatorially, an arrangement gives rise to an intersection lattice:

Definition 3 Let $\mathcal{A}$ be an arrangement of hyperplanes. The *intersection lattice* $L(\mathcal{A})$ is defined as the set of all possible intersections of hyperplanes in $\mathcal{A}$:

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{A}'} H \mid \mathcal{A}' \subseteq \mathcal{A} \right\}.$$

Partially ordering $L(\mathcal{A})$ by reverse inclusion yields a partially ordered set (poset).

The poset $L(\mathcal{A})$ is in fact essentially a *matroid*. In general, a matroid is a combinatorial abstraction of linear independence in vector spaces and forests in graphs. Since we will be exclusively working with rank 3 arrangements, we refrain from giving a general definition of a matroid and content ourselves with defining rank 3 matroids:

Definition 4 A matroid M of rank 3 is a bipartite graph[1] connecting atoms and coatoms such that every pair of atoms is jointly adjacent to exactly one coatom.

A matroid is called *representable over the field* k if there exists a rank 3 arrangement $\mathcal{A}$ such that there is a bijection between $L(\mathcal{A}) \setminus \{\emptyset \cup V\}$ and M respecting the poset structure such that hyperplanes and lines are mapped to atoms and coatoms, respectively.

The condition on a matroid demanding that every pair of atoms meets in exactly one coatom stems form the fact that every pair of hyperplanes defines exactly one line, i.e. 1-dimensional subspace.

Example 2 (continued) The intersection lattice of the braid arrangement $\mathcal{A}_3$ is depicted in Figure 2.

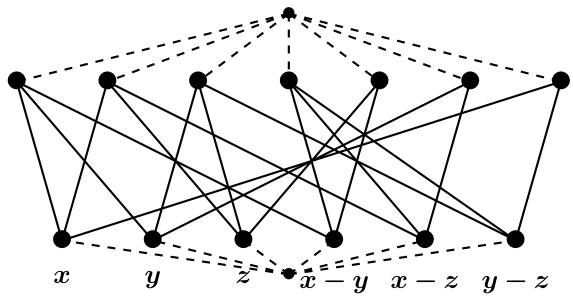

Figure 2: *The intersection* $L(\mathcal{A}_3)$.

Derivation module of an arrangement

Motivated from singularity theory, Saito introduced the notion of a logarithmic derivation of a hypersurface. Terao subsequently pioneered the study of such logarithmic derivation in the special case of hyperplane arrangements [Ter80].

Definition 5 Let $\mathcal{A}$ be an arrangement of hyperplanes in a vector space V over a field k of dimension ℓ. For every $H \in \mathcal{A}$, fix some linear defining equation $\alpha_H \in S := k[x_1 \ldots, x_\ell]$. The S-module of *logarithmic derivations* $D(\mathcal{A})$ is defined as

$$D(\mathcal{A}) := \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\},$$

where $\operatorname{Der} S$ is the free S-module of all derivations of S.

The arrangement $\mathcal{A}$ is called *free* if $D(\mathcal{A})$ is a free S-module, that is, it admits a basis over S.

The module $\operatorname{Der} S$ is a free module over S generated by all partial derivatives $\partial_{x_1}, \ldots, \partial_{x_\ell}$.

Example 2 (continued) The module of logarithmic derivations $D(\mathcal{A}_3)$ of the braid arrangement $\mathcal{A}_3$ is free with a basis

$$\begin{aligned}\theta_1 &:= x\partial_x + y\partial_y + z\partial_z,\\ \theta_2 &:= x^2\partial_x + y^2\partial_y + z^2\partial_z,\\ \theta_3 &:= x^3\partial_x + y^3\partial_y + z^3\partial_z.\end{aligned}$$

It is easy to verify that these derivations are in fact in $D(\mathcal{A}_3)$. We omit further arguments at this point which prove that these derivations are S-independent and in fact generate $D(\mathcal{A}_3)$.

Terao generalized this example by proving that in fact all reflection arrangements turn out to be free. Over the years, remarkable connections were established between the freeness of an arrangement and related arrangement properties. It is however fair to say that it is still quite mysterious to understand which arrangements are free and which are not. This is manifested in the main open conjecture in this area formulated by Terao about 40 years ago:

Conjecture 6 (Terao's freeness conjecture) The freeness of an arrangement $\mathcal{A}$ defined over a fixed field k depends only on its intersection lattice $L(\mathcal{A})$, that is its underlying matroid.

In other words, for two arrangements $\mathcal{A}$ and $\mathcal{A}'$ both defined over the same field k with $L(\mathcal{A}) = L(\mathcal{A}')$ Terao's conjecture asserts that $\mathcal{A}$ is free if and only if $\mathcal{A}'$ is free.

Our strategy to investigate Conjecture 6 in dimension 3 consists of the following three steps:

1. We generate all possibly relevant matroids in some size-range and save them in a database.

2. For each such matroid we determine the (possibly empty) *moduli space* of all representations in a way that can be easily specialized to any field.

3. We compute the nonfree locus within the moduli space of each relevant matroid. That is, all arrangements representing one fixed matroid that are nonfree. A matroid with a nonfree locus that is a proper subset of the moduli space (after specializing to some field) would be a counter-example to Conjecture 6.

A matroid database

Given an arrangement $\mathcal{A}$, one can consider the so-called *characteristic polynomial* $\chi(\mathcal{A}, t)$ which is a combinatorial invariant, i.e., it only depends on the intersection lattice $L(\mathcal{A})$. A celebrated theorem of Terao states that for a free arrangement, the characteristic polynomial $\chi(\mathcal{A}, t)$ factors over the integers [Ter81]. In particular, it suffices to investigate Terao's freeness conjecture for those arrangements, and hence for those matroids with integrally splitting characteristic polynomial.

[1] A bipartite graph is a graph whose vertices can be partitioned into two sets U, V such that every edge joins a vertex in U to one in V.

This result is the starting point for our matroid enumeration efforts. Using a parallelized algorithm in HPC-GAP, the high-performance computing version of GAP, we generated all 815107 simple rank 3 matroids with integrally splitting characteristic polynomial with up to $n = 14$ atoms. The code used for this computation is part of the GAP package MatroidGeneration [BK20]. We stored them in a public ArangoDB database, which can be accessed here: [BK19b]. Figure 3 shows a screen shot of this database.

As a second application, we found through the database that there is exactly one representable rank 3 matroid that is divisionally free but not inductively free with up to 14 atoms, cf. [BK19c].

We invite the reader to use the database for their own matroid-related experiments. We computed several relevant properties of the matroids such as the moduli space, the automorphism group, the Tutte polynomial, and several notions of combinatorial freeness. We are happy to assist researchers interested in including other properties of the matroids for potentially different applications.

How to test matroid representability?

The space of *all* representations (over some unspecified field k) of a rank 3 matroid M is an *affine variety*, namely an affine subvariety $V(I) \subseteq \mathbb{A}_{\mathbb{Z}}^{3n+1}$, where n is its number of atoms.

More precisely, let $\mathbb{A}_{\mathbb{Z}}^{3n+1} := \operatorname{Spec} R[d]$, where $R := \mathbb{Z}[a_{ij} \mid i = 1, \ldots, 3, j = 1, \ldots, n]$ and d a further indeterminate. To describe the ideal I' set $A := (a_{ij}) \in R^{3 \times n}$. For a subset $S \subseteq [n] := \{1, \ldots, n\}$ denote by A_S the submatrix of A with columns in S. Further, let $\mathcal{B}(M)$ be the set of triples in $[n]$ that are not contained in some coatom of M (these are the bases of M in classical matroid terminology). Then

$$I := \langle \det(A_D) \mid D \subseteq [n], |D| = 3, D \notin \mathcal{B}(M) \rangle + \left\langle 1 - d \prod_{B \in \mathcal{B}(M)} \det(A_B) \right\rangle \trianglelefteq R[d].$$

It follows that M is representable (over some field k) if and only if $1 \notin I$. This ideal membership problem can be decided by computing a Gröbner basis of I. This is basically the algorithm suggested in [Oxl11].

To make computations feasible, it is an essential step to embed the moduli space $V(I)$ into some smaller dimensional affine space. We used the GAP package ZariskiFrames [BKLH19] to compute such an embedding.

Application to Terao's conjecture

To investigate Terao's freeness conjecture it suffices to consider matroids that satisfy the following conditions:

- They are representable over some field: If a matroid does not admit any representation there is nothing to check.
- They are not inductively free: Inductive freeness is a combinatorial condition on the matroid which ensures that all representations are free arrangements [Ter80].
- They are balanced: Unbalancedness is another combinatorial condition for which it is a priori known whether all representations are free or nonfree.
- They are not uniquely representable: If a matroid only admits a unique representation (up to a coordinate-change and field extensions) it also cannot admit a free and a nonfree representation.

Somewhat surprisingly, it turns out that there are only 9 rank 3 integrally splitting matroids of size up to 14 satisfying all of these conditions. They have 9, 11, 12, or 13 atoms. This already verifies Terao's freeness conjecture for rank 3 arrangements with up to 14 hyperplanes with the above 9 exceptions. Figure 3 shows a screen shot of the query in our database of the matroids satisfying the above conditions of size 14. As shown at the bottom of the figure, this query does not yield any matroid.

Subsequently, we computed the nonfree locus within the moduli space of the 9 remaining matroids. This locus is a closed subvariety of the affine moduli space which we computed using Fitting ideals in GAP using SINGULAR.

We found that over a fixed field all representations are either all free or all nonfree. As previously observed by Ziegler [Zie90], we found new examples of matroids which admit free and nonfree representations over different fields. A new example of this type is the matroid underlying the so-called pentagon arrangement depicted in Figure 4. Projectively, this arrangement consists of the five sides of a regular pentagon, together with its five diagonals and the line at infinity. In [OT92] it is explained that this arrangement is in fact free. Our methods prove that any arrangement representing the same matroid over a field of characteristic not 2 is free. However, the matroid is also representable over fields of characteristic 2 containing the field $\mathbb{F}_4$ with 4 elements and all of these representations are nonfree.

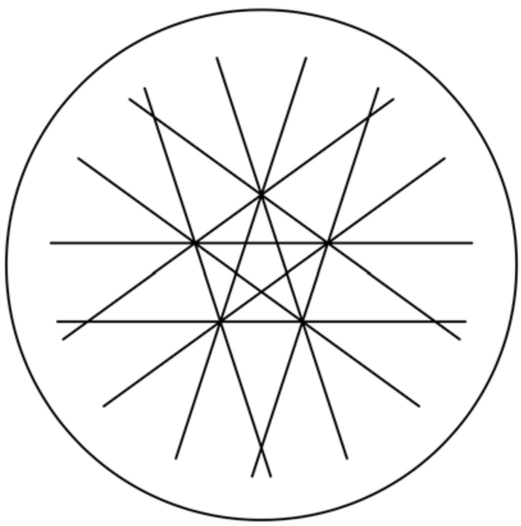

Figure 4: *The pentagon arrangement.*

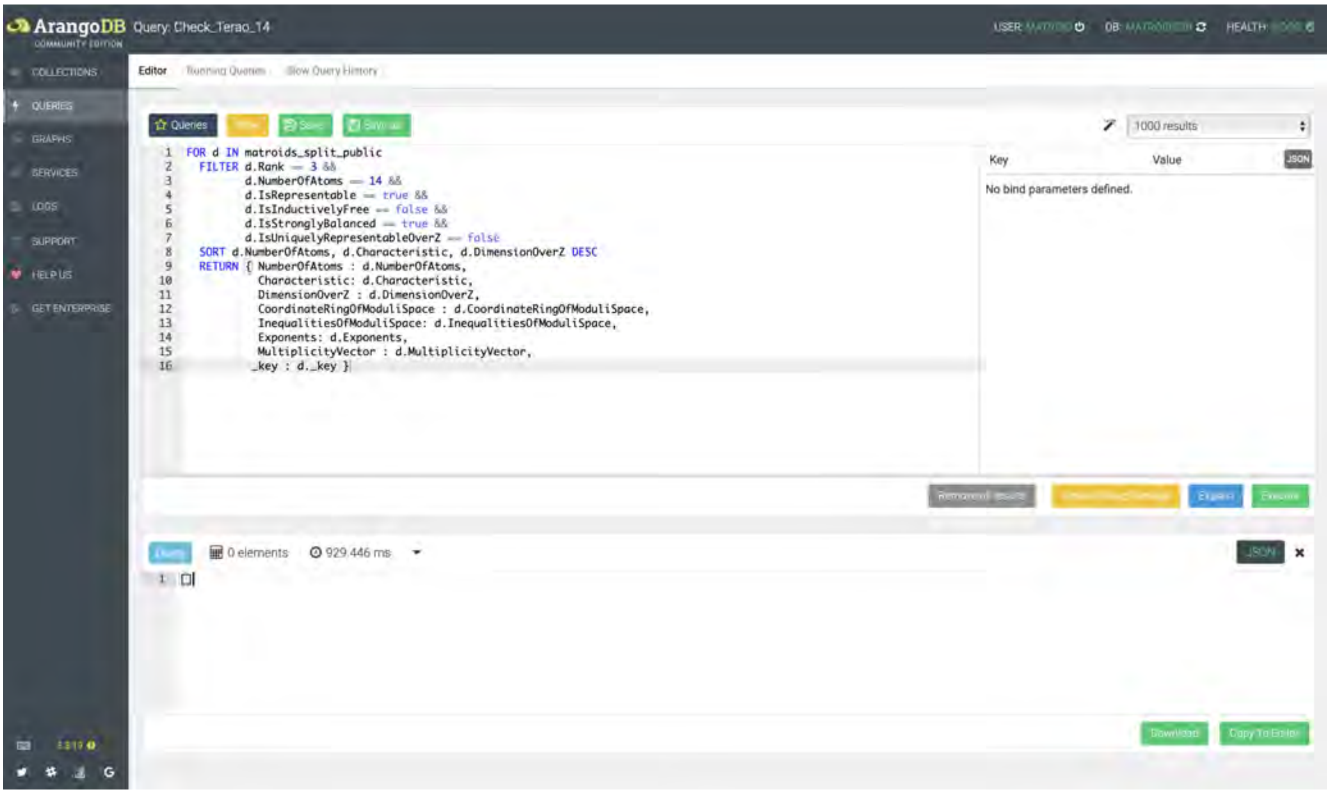

Figure 3: *A screenshot of the matroid database showing the query that proves Terao's freeness conjecture for rank* 3 *arrangements with* 14 *hyperplanes.*

In total we obtained the following main result:

Theorem 7 Terao's freeness conjecture is true for arrangements in dimension 3 with up to 14 hyperplanes in arbitrary characteristic.

Previously, Dimca, Ibadula, and Măcinic verified Terao's conjecture for arrangements with up to 13 hyperplanes in $\mathbb{C}^3$.

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