Symmetric embeddings of the icosahedron

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Introduction

Among the well-known Platonic solids is the icosahedron whose 20 faces are equilateral triangles of the same congruence type. At each of the twelve vertices five triangles meet and form what we call an umbrella. This convex polyhedron is only one geometric incarnation of an abstract icosahedron that admits variations of the Platonic solid as follows. Prescribing the incidence structure and the Euclidean geometry of its flat faces leaves – as it turns out – infinitely many possibilities of realizing the icosahedron as a surface made up of equilateral triangles in three-dimensional Euclidean space $\mathbb{R}^3$. Most of these polyhedra are self-intersecting.

Some non-convex embeddings can be obtained from the convex one as follows. For any vertex V of the regular icosahedron the five neighbouring vertices lie in a plane. Orthogonal reflection of V through the plane yields a "dented icosahedron". Figure 5 shows an icosahedron that can be obtained from the regular one by denting at two vertices at combinatorial distance 3.

In recent work [7] by Brakhage, Niemeyer, Plesken, Strzelczyk and the author of this note, the embeddings of the icosahedron admitting non-trivial symmetry have been classified. (The restriction to non-trivial symmetry was decided upon after it had turned out that the cases with trivial symmetry would be too numerous to deal with in the same approach.) Here we summarize our results and give a few details on how the classification was obtained. For more details we refer to the paper [7] as well as its accompanying web page.

There is a vast literature on polyhedra, cf., e.g., [2], [10], [15], [16], to cite just a few of the most prominent ones. The problem addressed here can also be understood as the problem of realizing a given graph in $\mathbb{R}^3$ with prescribed edge lengths and is linked with questions about rigidity, cf., e.g., [14]. Even closer related to the topic of this note, e.g., the existence of icosahedra all of whose faces are congruent to one scalene triangle was shown in [12].

Simplicial surfaces

The context in which said classification has been achieved is a more general study of surfaces that are composed of triangles, which we call *simplicial surfaces*. Motivated by a question posed by an architect at RWTH Aachen University what would be the types of surfaces one could build from a collection of congruent triangles, a systematic mathematical investigation has been launched. A first monograph on the subject is in preparation [13].

Simplicial surfaces are now studied from different points of view:

- considered as combinatorial objects, simplicial surfaces can be generated and classified by group-theoretical methods;
- considered as (abstract) geometrical objects, simplicial surfaces can be studied as manifolds with singularities;
- embeddings of simplicial surfaces in Euclidean space form a rich source of concrete mathematical objects of interest in algebraic geometry and related fields as well as for applications in the sciences and engineering.

Embeddings of an infinite family of simplicial surfaces with dihedral symmetry, called double $2n$-gons, have been studied in [8].

Quite a few results on combinatorial aspects have already been obtained, which are beyond the scope of this note. As a byproduct a package `SimplicialSurfaces` [1] for GAP [11] has been developed by Markus Baumeister, Alice C. Niemeyer and further collaborators. For instance, it allows to generate all (combinatorial) simplicial surfaces for a given number of faces (up to isomorphism), possibly taking further given conditions on the incidence structure into account. It also provides routines for classifying edge colorings of these surfaces, which is relevant for embeddings realizing a single congruence type of triangle.

The topic of this note is an intriguing case study concerning the third aspect of our research program listed above. It turned out that, in the order of increasing complexity of simplicial surfaces, the icosahedron is a first challenging case to analyze.

Figure 1: *An icosahedron with symmetry group ${C_2}^2$ from the first row of Table 2*

The icosahedron

We may use the automorphism group of the (combinatorial) icosahedron to give a succinct definition of its incidence structure, involving the vertices numbered from 1 to 12, edges represented by unordered pairs of vertices, and faces represented as 3-element sets of vertices.

We choose the permutations of twelve points

$$\begin{aligned} a &:= (1,2)(3,4)(5,7)(6,8)(9,11)(10,12), \\ b &:= (1,10)(3,9)(2,12)(4,11)(5,6)(7,8), \\ c &:= (1,7)(2,3)(4,11)(5,12)(6,8)(9,10), \\ d &:= (1,12)(3,9)(2,10)(4,11)(5,7)(6,8) \end{aligned}$$

as a generating set for the combinatorial automorphism group $A \cong C_2 \times A_5$ of the icosahedron. The center of A is generated by the involution d, whose effect on the (embedded) regular icosahedron with barycenter $(0,0,0)$ would be the point reflection. According to the above choice, the combinatorial description of the icosahedron is endowed with a labeling: the orbit of 1 under A consists of the vertices $1, 2, \ldots, 12$; the orbit of $\{1,2\}$ is the set of 30 edges; and the orbit of $\{1,2,3\}$ is formed by the 20 faces. Thus d interchanges combinatorially opposite vertices.

Symmetric embeddings

Our task is to classify the embeddings of the icosahedron in $\mathbb{R}^3$ realizing the edges as line segments of length 1, with 12 distinct vertices, and admitting a non-trivial symmetry group. Such an embedding is given by a map $p : \{1,\ldots,12\} \to \mathbb{R}^3$ such that the Euclidean distance between $p(i)$ and $p(j)$ is equal to 1 for all edges $\{i,j\}$ of the icosahedron, and we require that such a map be injective. We usually arrange $p(1),\ldots,p(12)$ as columns of a *coordinate matrix* $M \in \mathbb{R}^{3\times 12}$.

We are thus interested, in principle, in finding the real solutions of a system of 30 multivariate quadratic equations in $12 \times 3 = 36$ unknowns (which, however, do not incorporate all the above conditions, e.g., vertex-faithfulness, symmetry). Since this system of polynomial equations is very difficult to solve, our approach had to be refined.

Figure 2: *Convex icosahedron with symmetry group $C_2 \times A_5$ from the second row of Table 2*

Of course, the group $E(3) \cong \mathbb{R}^3 \rtimes O(3,\mathbb{R})$ of Euclidean isometries acts on the set of embeddings, and we factor out this action as follows. Translations allow to move the barycenter of the (equally weighted) points $p(1), \ldots, p(12)$, i.e. the columns of M, to the origin. Now, by considering the *Gram matrix* $G := M^{tr}M$ we also factor out the action of the orthogonal group $O(3,\mathbb{R})$. Note that a Gram matrix G is a real symmetric 12×12 matrix that is positive semidefinite of rank at most 3.

As a result for our classification task, the isometry classes of (embedded) icosahedra are given by the equivalence classes of Gram matrices that are defined with respect to conjugacy by permutation matrices.

The group A acts on the set of Gram matrices G by simultaneous row and column permutations, and the stabilizer of G in A is the *symmetry group* of the (embedded) icosahedron.

Theorem 1 The subgroups U of A with more than one element that arise as symmetry group of an icosahedron fall into 10 conjugacy classes of subgroups of A. For each of these conjugacy classes the number of (isometry classes of) icosahedra with symmetry group in that class is listed in Table 1.

Automorphism group $U \leq A \cong C_2 \times A_5$	Number of icosahedra
$C_2 \times A_5$	2
$C_2 \times D_{10}$	4
$C_2 \times D_6$	2
D_{10} $(\nleq A_5)$	3
D_6 $(\nleq A_5)$	2
${C_2}^2$ $(\ni d)$	1
${C_2}^2$ $(\not\ni d, \nleq A_5)$	5
${C_2}^2$ $(\leq A_5)$	1
C_2 $(\leq A_5)$	5
C_2 $(\not\ni d, \nleq A_5)$	10
C_2 $(= \langle d \rangle)$	∞

Table 1: *Equivalence classes of symmetric embeddings of the icosahedron*

[1]There is a discrete subset of points on the curve corresponding to degenerate icosahedra, in the sense that the set of vertices is mapped to fewer than 12 points.

The last row of Table 1 corresponds to an algebraic curve, almost all[1] of whose (closed) points correspond to icosahedra with (at least) the specified symmetry. Hence, there is a continuous family of such icosahedra, which can be understood as a *flexible icosahedron*, i.e., an icosahedron admitting motions preserving all edge lengths, but which are not rigid motions.

By the following famous theorem, there is only one convex icosahedron and it is necessarily rigid.

Theorem 2 (Cauchy's Rigidity Theorem, 1813, [9]) If two closed *convex* polyhedra are combinatorially equivalent and corresponding pairs of faces of these two polyhedra are congruent, then the two polyhedra are congruent (i.e., there exists a rigid motion transforming one to the other).

Figures 1–7 show visualizations of some of the icosahedra with non-trivial symmetry listed in Table 2, the details of which we explain in the next section.

For more visualizations and, in particular, animations of the flexible icosahedron found in the last row of Table 1 we refer to the web page accompanying our paper [7].

Figure 3: *Non-convex icosahedron with symmetry group $C_2 \times A_5$ from the third row of Table 2*

Classification

Since we focused on icosahedra with non-trivial symmetry group, we went through a list of representatives of conjugacy classes of minimal subgroups U of A, namely

$$\langle abc\rangle \cong C_3, \qquad \langle ac\rangle \cong C_5, \qquad \langle a\rangle \cong C_2,$$
$$\langle d\rangle \cong C_2, \qquad \langle ad\rangle \cong C_2,$$

and we determined the embeddings of the icosahedron whose symmetry groups contain U. We arrived at computationally tractable tasks thanks to the following consequence of the fact that any two coordinate matrices associated with the same Gram matrix G differ multiplicatively by an orthogonal matrix.

Lemma Let G be a Gram matrix of an icosahedron with symmetry group U. Then there exist a faithful orthogonal representation $\delta : U \to O(3, \mathbb{R})$ and a coordinate matrix $M \in \mathbb{R}^{3\times 12}$ such that

$$\delta(g)\, M = M\, \pi(g) \quad \text{for all } g \in U \tag{1}$$

and $G = M^{tr} M$, where $\pi : A \to \mathrm{GL}(12, \mathbb{R})$ is the representation of A by permutation matrices.

We were able to rule out the minimal subgroups of odd order and therefore had to investigate the three representatives U isomorphic to C_2, whose generators can be mapped to one of the following three matrices by δ:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence, we had to consider nine pairs of U and δ. In each case the imposed symmetry condition allows via (1) to drastically reduce the number of unknowns in the system of polynomial equations for the entries of M. We computed a primary decomposition of the corresponding ideal I in the appropriate polynomial ring R. In all but one case the Krull dimension of I turned out to be zero, in just one case the Krull dimension of I was one.

For a maximal ideal m arising in the minimal primary decomposition of a zero-dimensional ideal $I \trianglelefteq R$ as above we refer to the image modulo m of $G = M^{tr}\, M$ with reduced number of unknowns as *formal Gram matrix*. Several checks had to be performed to rule out the following scenarios:

- the field R/m may have no real embedding;
- the resulting formal Gram matrix may not be positive semi-definite of rank at most 3;
- U may be a proper subgroup of the symmetry group of the resulting formal Gram matrix.

Table 2 records the field degree $d_G := [R/m : \mathbb{Q}]$, the number $r_{1,G}$ of real embeddings of R/m, the number r_G of those real embeddings of R/m defining a positive semidefinite formal Gram matrix, and the number $r_{f,G}$ the number of those that moreover yield 12 pairwise distinct points $p(1), \ldots, p(12)$.

The rows in Table 2 are arranged in blocks following the order in which we investigated the nine cases. Three of the nine cases did not contribute to that list, and one case led to the ideal of Krull dimension 1 to be discussed further in the next section. Since the symmetry group for the resulting Gram matrix can strictly contain the chosen minimal subgroup U, duplicate formal Gram matrices arising from different cases had to be eliminated.

For computing in R/m we chose a primitive element, preferably the trace of the Gram matrix, if possible. However, it is not always the case that the trace generates R/m. Table 2 lists the minimal polynomial of the trace of G over Q in its last column.

Finally, in the second column of Table 2 in certain cases we indicate the trace value of the single generator, or the trace values of the two generators above a horizontal line and the trace value of their product below the horizontal line, where $+$, $-$, $\bullet$ stands for 1, -1, -3, respectively.

$S := \mathrm{Stab}_A$	$\mathrm{Syl}_2(S)$	d_G	$r_{1,G}$	r_G	$r_{f,G}$	Trace relation
${C_2}^2$	$\langle a,d\rangle \frac{\bullet -}{+}$	8	4	2	1	$\lambda^4 - \frac{76}{3}\lambda^3 + 238\lambda^2 - \frac{4964}{5}\lambda + \frac{23767}{15}$
$C_2 \times A_5$	$\langle a,b,d\rangle$	2	2	2	2	$\lambda^2 - 15\lambda + 45$
$C_2 \times D_{10}$	$\langle a,d\rangle \frac{- \bullet}{+}$	2	2	2	2	$\lambda^2 - 15\lambda + \frac{269}{5}$
${C_2}^2$	$\langle a,bd\rangle \frac{- +}{+}$	2	2	2	2	$\lambda^2 - \frac{71}{5}\lambda + \frac{10561}{225}$
$C_2 \times D_{10}$	$\langle a,d\rangle \frac{- +}{+}$	4	2	2	2	$\lambda^4 - 18\lambda^3 + \frac{583}{5}\lambda^2 - \frac{1658}{5}\lambda + \frac{9101}{25}$
$C_2 \times D_6$	$\langle a,d\rangle \frac{- +}{+}$	4	4	2	2	$\lambda^4 - 26\lambda^3 + 243\lambda^2 - 970\lambda + 1397$
${C_2}^2$	$\langle a,bd\rangle \frac{- +}{+}$	24	10	6	3	$\lambda^{12} - \frac{5179\cdot 2^2}{3^2\cdot 5^2}\lambda^{11} \pm \cdots$
${C_2}^2$	$\langle a,b\rangle \frac{- -}{-}$	30	18	6	1	$\lambda^5 - \frac{117}{2}\lambda^4 \pm \cdots$
C_2	$\langle a\rangle$ $-$	172	48	20	5	$\lambda^{43} - \frac{73\cdot 7\cdot 11\cdot 461687}{2^2\cdot 3^3\cdot 5^2\cdot 29\cdot 79}\lambda^{42} \pm \cdots$
D_{10}	$\langle ad\rangle$ $+$	2	2	2	2	$\lambda^2 - \frac{44}{3}\lambda + \frac{2131}{45}$
D_6	$\langle ad\rangle$ $+$	2	2	2	2	$\lambda^2 - \frac{68}{5}\lambda + \frac{1111}{25}$
C_2	$\langle ad\rangle$ $+$	36	12	8	4	$\lambda^{18} - \frac{1106}{9}\lambda^{17} \pm \cdots$
C_2	$\langle ad\rangle$ $+$	168	40	24	6	$\lambda^{42} - \frac{2\cdot 719\cdot 1223}{3^3\cdot 5\cdot 43}\lambda^{41} \pm \cdots$
D_{10}	$\langle ad\rangle$ $+$	4	2	2	1	$\lambda^2 - \frac{26}{3}\lambda + \frac{149}{9}$

Table 2: *List of all formal Gram matrices with non-trivial symmetry*

Figure 4: *An icosahedron with symmetry group $C_2 \times D_{10}$ from the third row of Table 2*

A curve of icosahedra

If $U = \langle d\rangle$ and $\delta(d)$ is equal to the first orthogonal matrix listed earlier, the ideal I is of Krull dimension 1. In this case computational resources did not allow to compute a primary decomposition of I. We managed to prove the existence of a flexible icosahedron in this case by recognizing the corresponding curve as integral curve of a polynomial vector field. Knowing the coordinates of some point on the curve in terms of algebraic numbers from another case of the classification as well as the tangent vectors to the curve, we can solve (at least approximately) the corresponding initial value problem. Moreover, standard theorems on ordinary differential equations yield the existence of $\varepsilon > 0$ and a non-constant real analytic map $\Phi : [0,\varepsilon) \to \mathbb{R}^{3\times 12}$ such that $\Phi(t)$ is the coordinate matrix of a $\langle d\rangle$-invariant icosahedron for all by finitely many $t \in [0,\varepsilon)$. Since the map $t \mapsto \Phi(t)^{tr}\,\Phi(t)$ is not constant and only finitely many isometry types of icosahedra correspond to a Gram matrix $\Phi(t)^{tr}\,\Phi(t)$, it follows that there are infinitely many pairwise inequivalent $\langle d\rangle$-invariant icosahedra.

Figure 5: *An icosahedron with symmetry group $C_2 \times D_{10}$ from the third row of Table 2*

Figure 6: *An icosahedron with symmetry group $C_2 \times D_6$ from the sixth row of Table 2*

Computational aspects

In a first approach to solving the system of multivariate quadratic equations expressing that the 30 edges of the embedded icosahedron have length 1 we applied the software Bertini [3] implementing homotopy continuation of solutions to systems of polynomial equations. It gave a very good impression on how many solutions to expect for our problem, but we were not able to convert the results obtained into exact solutions nor to make progress in the classification task with that information.

For the new successful attempt with exact arithmetic we used Magma [6] for computing primary decompositions of polynomial ideals and the Maple package Involutive [4] and GINV [5] for further processing of the prime ideals.

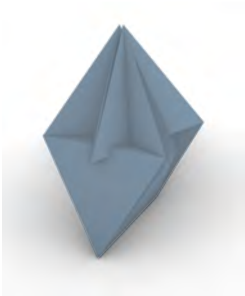

Figure 7: *An icosahedron with symmetry group $C_2{}^2$ from the seventh row of Table 2*

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