On the Convergence of a Leader-Following Discrete-Time Consensus Protocol

Sunantha Sodsee[1,2], Maytiyanin Komkhao[1], Zhong Li[1]
Wolfgang A. Halang[1], Phayung Meesad[2]

Faculty of Mathematics and Computer Science
Fernuniversität in Hagen, Germany[1]
Faculty of Information Technology
King Mongkut's University of Technology North Bangkok, Thailand[2]
Email: sunantha.sodsee@fernuni-hagen.de

Abstract: A discrete-time consensus protocol with the leader-following control is proposed, which is for agents to follow either static or time-varying reference state of the leader. An agent updates its state based on only the current information available from its neighbors. To estimate the consensus convergence, the eigenvalues are investigated by the Gershgorin's theorem, and simulations will be given to show the effectiveness of the proposed consensus protocol.

Keywords: Discrete-time consensus, Leader-following, Gershgorin's theorem, Static and time-varying state.

1 Introduction

Consensus problem has a long history in computer science, particularly in distributed computing [Ly97][De74]. However, in the context of networked multi-agent systems, consensus problems is related to the group coordination, which is for a group of agents to make a decision or to reach an agreement regarding a certain quantity of interest that depends on the states of all agents. The consensus problem is also related to the cooperative control problem, in which the global objective is to reach a consensus of all agents [SFS07]. Under a certain topology of the multi-agent systems, the consensus problem is to design the consensus algorithm or protocol, or say, the interaction rule, that specifies the information exchange between the agents on the network, so as for the agents to reach a consensus [FM04][OFM07][LC08][RBA05].

In 1995, Vicsek first proposed a discrete-time consensus model, and simulated the phase transitions in a self-driven particles [Vi95]. Then, Jadbabie *et al.* provided a theoretical explanation for Vicsek's observation by proposing the nearest neighbor rules to coordinate the group behavior of mobile autonomous agents [JLM03]. Fax and Murray [FM04] considered the information flow and cooperative control for unmanned vehicles. Ren and Beard [RB05] studied the consensus seeking in multi-agent systems under dynamically changing interaction topologies. Olfati-Saber and Murray proposed the consensus protocols for the networks of dynamic agents [OM04][OM03]. So far, most consensus protocols just guarantee the agents to converge to an emergent consensus value, but not to reach a specific value [FXD08][KB06][XWW06][RB08]. For this purpose, leader-following consensus models have been proposed [Re07][CRL09][SK08][HH07][Li08], where an external control signal acting as the leader who is connected to agents is used to drive the group behavior of the system, and a follower updates its state based on the previous or current information available from its neighbors and the leader, so as for the agents/followers to reach a consensus associated with the leader's state.

The leader-following continuous-time consensus model was proposed in the form of [RB08]

$$\dot{\xi}_i = -\sum_j a_{ij}(\xi_i - \xi_j) - a_{i(n+1)}(\xi_i - \xi^r), \qquad (1)$$

where ξ_i represents the state of follower i (i=1,2,…,n), the leader is labelled as n+1, ξ^r is the reference state acting as the state of the leader, and $(a_{ij})_{n\times n}$ is the adjacency coefficient matrix and $a_{i(n+1)} \equiv 1$ for all i if each follower is connected with the leader. With this model, the followers can track the static state of the leader but fail with the time-varying state of the leader. The corresponding discrete-time consensus model was then proposed and is called P-like discrete-time consensus algorithm [CRL09], which is described as following,

$$\xi_i[k+1] = \xi_i[k] - T\sum_j a_{ij}(\xi_i[k] - \xi_j[k]) - Ta_{i(n+1)}(\xi_i[k] - \xi^r[k]), \qquad (2)$$

where T is the sampling period, $\xi_i[k]$ and $\xi^r[k]$ are the state of follower i and the state of the leader at time step k, respectively.

To determine the stability properties of the consensus protocol, the location of the eigenvalues of the network is concerned. Which the second smallest eigenvalue of Laplacian matrix [OFM07] or the second largest eigenvalue of Perron matrix [OFM07] of the network topology is a measure of performance or speed of consensus algorithms. As well the Gershgorin’s theorem is an one way to estimate the eigenvalues for finding the trace of the network matrix [Br07].

Normally, the reference state of the target can be static or changed dynamically (time-varying). Therefore, the question of how to develop a consensus protocol so as for the agents to follow both static and time-varying state of the leader needs to be addressed; and how to analyze the convegence speed of the proposed protocol are the main concern of this paper. A leader-following discrete-time consensus protocol is first proposed in this paper, with which the agents can follow both the static and time-varying state of the leader and the follower updates its state by using only the current exchanged information from its neighbors and the leader; As well as, the Gershgorin’s theorem is applied to estimate the eigenvalues on this proposed consensus protocol.

The rest of this paper is organized as follows. Background and preliminaries are introduced in Sec.II. Sec.III presents the proposed model and the numerical simulation result is presented in Sec.IV. Finally, the work is concluded in Sec.V.

2 Background and Preliminaries

An interactive topology of network of agents is represented by a directed graph $G_n = (V_n, E_n)$, where V_n is the set of vertices v_i, i={1,2,…,n},and E_n the set of edges $\rho_{ij} = (v_i, v_j), i, j = \{1,2,...,n\}$. Herein, the edge from v_j to v_i donates that v_i receives information from v_j. The adjacency matrix $A_n = [a_{ij}] \in R^{n\times n}$ associated with G_n is defined as $a_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \in E_n, \\ 0, & \text{Otherwise.} \end{cases}$. The directed G_n is called balanced if $\sum_{i\neq j} a_{ij} = \sum_{i\neq j} a_{ji}$ for all $i, j \in V_n$. $N_i\{v_j | a_{ij} \neq 0 \text{ and } i \neq j\}$ is the set of neighbors of v_i, and $|N_i|$ the number of neighbors of v_i or in-degree of node i, $\deg_{in}(v_i) = \sum_j a_{ij}$. The degree matrix of digraph G_n is a diagonal matrix $D_n = [d_{ij}]$ where $d_{ii} = \sum_{i\neq j} a_{ij}$ or $\deg_{in}(v_i)$. Further, the graph Laplacian L_n is defined as $L_n = D_n - A_n$. It is It is obvious that all row-sums of L_n are zero, hence, L_n always has a zero eigenvalue λ_1=0.

The original discrete-time consensus protocol is describe by [OFM07]

$$x_i[k+1] = x_i[k] - \varepsilon \sum_{j\in N_i} a_{ij}(x_i[k] - x_j[k]), \tag{3}$$

where $x_i[k]$ denotes the state of agent i at time step k, and $0 < \varepsilon \leq 1/\Delta$ is the sampling period, in which Δ is the maximum degree of agents.

Further, Eq.3 can be recast as $x[k+1]=P_n x[k]$, where $P_n=I_n- \varepsilon L_n$ is the Perron matrix of graph G_n, and I_n is the identity matrix. Assume that P_n is a primitive-nonnegative matrix, and denote w as the nonnegative left eigenvector associated with eigenvalue 1, *i.e.*, $w^T P_n = w^T$.

A group of agents is said to reach a global consensus if $x_j[k] = x_i[k]$ for each pair (i,j),~i,j =1, 2,…, n and $i \neq j$, and the common agreement value of all agents is called the group decision value, denoted by $\alpha = \sum_i w_i x_i[0]$, where w_i is the left eigenvalue associated with eigenvalue $\lambda_i (i = 1,2,...,n)$, satisfying $\sum_i w_i = 1$, and $x_i[0]$ is the initial state [OFM07][FXD08][KB06]. For a balanced digraph, one has $w = (\frac{1}{n})\,\mathbf{1}$, $\alpha = (\frac{1}{n})\,\mathbf{1}^T x_i[0]$, and

$$\lim_{t \longrightarrow \infty} \alpha(t) = \frac{\sum_i x_i[0]}{n}, \tag{4}$$

where $t = k\varepsilon$ [OFM07][KB06].

3 Leader-following Control

In this section, the leader-following discrete-time consensus algorithms with the time-varying leader state is concerned.

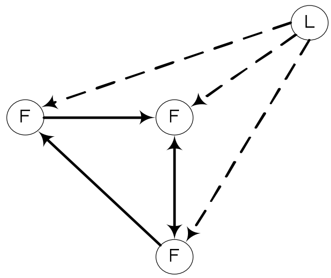

Figure 1: Leader-follower interaction topology

A leader is added to the system, and is connected to all followers. Denote the followers as $x_i^f, i = \{1,2,..,n\}$ and the leader as x_{n+1}^l or x^l. The multi-agent system can be represented as the directed graph or digraph $G_{n+1} = (V_{n+1}, E_{n+1})$, where $V_{n+1} = \{v_1, v_2, ..., v_{n+1}\}$ is the set of nodes and $E_{n+1} \subseteq V_{n+1} \times V_{n+1}$ is the set of edges.

Here, only the leader sends the information to the followers. Fig.1 depicts the interaction topology between the leader and the followers. The dash line connecting between the leader and follower means that the leader sends the information to the follower. In additional, the thick line presents the communication between followers. Because the reference state of leader can be static or time-varying, a discrete-time consensus protocol is given as

$$x_i^f[k+1] = x_i^f[k] - a_{i(n+1)}(x_i^f[k] - x^l[k]) - \varepsilon \sum_j a_{ij}(x_i^f[k] - x_j^f[k]), \qquad (5)$$

Let $\varepsilon = \frac{1}{n+1}$. It is obvious that $a_{i(n+1)} \equiv 1$, $i = \{1,2,...,n\}$. The consensus is said to be reached if $x_i^f[k] = x^l$ as $k \longrightarrow \infty$. (5) can be recast in the matrix form

$$x^f[k+1] = (P - B)x^f[k] + bx^l[k], \qquad (6)$$

where the matrix $P = I - \varepsilon L$ is the Perron matrix, $B = diag[a_{1(n+1)}, a_{2(n+1)}, ..., a_{n(n+1)}]$, and the vector $b = [a_{1(n+1)}, a_{2(n+1)}, .. a_{n(n+1)}]^T$. (6) can be further written in the compact form

$$x^f[k+1] = Fx^f[k] + bx^l[k], \qquad (7)$$

where *F*=*P*-*B*.

An irreducible stochastic matrix *P* is primitive if it has one singular eigenvalue with maximum modulus. *P* is a row stochastic non-negative matrix; row-sums of *P* are 1, hence *P* always has a one eigenvalue $\lambda_1 = 1$. The convergence analysis of the discrete-time consensus algorithm relies on the second largest eigenvalue of P. It reachs with a speed that is faster or equal to $\lambda_2 = 1 - \varepsilon\mu_2(L)$, where $\mu_2(L)$ is the second smallest eigenvalue of L and λ_2 is the second largest eigenvalue of P [OFM07]. The set of eigenvalues of P can be ordered sequentially as $1 = \lambda_1 > \lambda_2 \geq ... \geq \lambda_{n+1}$. According to the Gershgorin's theorem [Br07], all eigenvalues of the network matrix (P) are located on a unit circle. Each eigenvalue lies within the Gershgorin disc centered at (r_i,0), where r_i is the P_{ii}. The radius of each disc is calculated by $d_i = \sum_{i \neq j} | P_{ij} |$.

Theorem 1

All eigenvalues of P must lie in a unit circle centered at (0,0).

Proof

A leader is connected to all followers in the system, and only the leader sends the information to the followers (there is no feedback information from the followers). The eigenvalues of followers are $0 < \lambda_i < 1$ and these eigenvalues lie in the Gershgorin disc at (r_i,0), where $0 < r_i < 1$ and the radius $0 < d_i < 1$. On the other hand, the eigenvalue of the leader is always one, it is centered at (r_{n+1},0) , where r_{n+1}=1 then the radius d_{n+1} is zero.

4 Numerical Simulations

In this section, we present an example, the unbalanced digraph, to show the effectiveness of the proposed work.

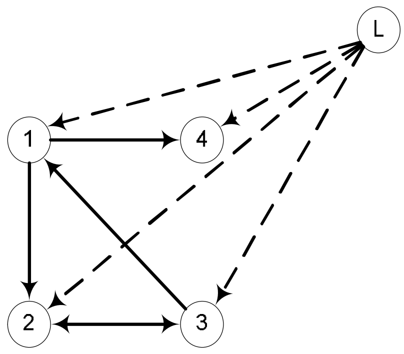

Figure 2: Example of unbalanced digraph

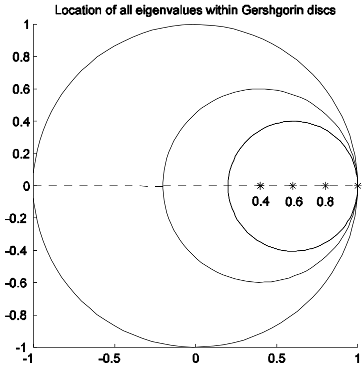

Figure 3: Location of eigenvalues

Fig. 2 depicts the example of unbalanced digraph [RB08], there is one leader connected to four followers. This interaction graph corresponds to the Parron matrix as follow

$$P = \frac{1}{5}\begin{bmatrix} 3 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix},$$

The eigenvector λ of P is $\lambda = [0.6 \quad 0.8 \quad 0.4 \quad 0.4 \quad 1]^T$. Fig.3 shows the Gershgorin discs of all eigenvalues. The centers of each disc are (0.6,0), (0.4,0), and (0,0). The bigest disc is a unit circle.

To estimate the location of eigenvalues, followers' eigenvalues are 0.6, 0.8, 0.4, and 0.4. They are located in Gershgorin discs centered at (0.6,0), (0.4,0), (0.6,0), and (0.6,0). Their discs' radius are 0.4, 0.6, 0.4, and 0.4 respectively. On the other hand, the eigenvalue of the leader is one, it is centered at (1,0). Thus, the location of all eigenvalues are within the unit circle that centered at (0,0), as well as they are located within every Gershgorin disc.

To show the effectiveness of the proposed protocol, the initial state of followers is $x^f[0] = [-0.5 \quad 0 \quad 0.5 \quad 1.5]^T$ and the reference state of the leader are static with $x^l = 1$ and time-varying with $x^l = \cos(t)$. Presenting the global consensus, Fig. 4 and 5 are depicted the simulation results that the followers can reach the consensus following the static state of the leader (Fig.4) and the time-varying state of the leader (Fig.5), respectively.

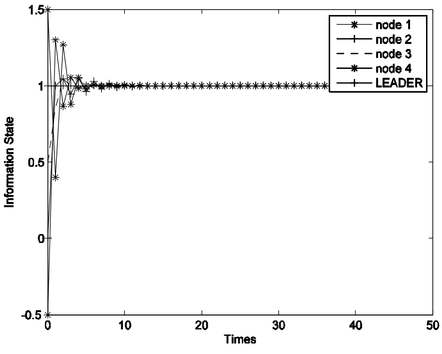

Figure 4: Static state of leader

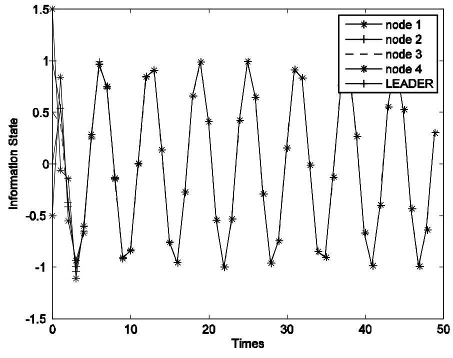

Figure 5: Time-varying state of leader

5 Conclusions

In this paper, the discrete-time consensus protocol based on a leader-following behavior of multi-agent system has been proposed, suitable for both the static and time-varying states of the leader. It uses only the current state of agents to identify the updated state of followers, which reduces the memory storage. The simulation result has shown that this proposed protocol is efficient. The states of followers can converge to the consensus following the leader's state, which their eigenvalues location analyzed by the Gershgorin’s theorem.

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